# Math 165: Homework 3

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, March 13th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

**Exercise 1.** Determine if the following sequences of functions converge and, if so, say whether this convergence is pointwise or uniform on the given set $E$. Also, identify the limit function. Please prove your assertions.

1. $f_n(x) = x/n$ on $E = [a, b]$ for fixed $-\infty < a < b < \infty$.
2. $f_n(x) = x/n$ on $E = [0, \infty)$.
3. $f_n(x) = e^{x/n}$ for $E = [0, 1]$
4. $f_n(x) = e^{x/n}$ for $E = [0, \infty)$.
5. $f_n(x) = \frac{x}{n} e^{-x/n}$ for $E = [0, \infty)$.

**Exercise 2.** Prove that the following limits exist and evaluate them:

1. $$\lim_{n\to\infty} \int_{-1}^{1} e^{x^2/n}\, dx$$
2. $$\lim_{n\to\infty} \int_{0}^{3} \frac{nx^2+3}{x^2+nx}\, dx$$
3. $$\lim_{n\to\infty} \int_{0}^{\pi/2} \sqrt{\sin(x/n)+\cos(x/n)}\, dx$$
4. $$\lim_{n\to\infty} \int_{0}^{\pi} e^{ix/n}\, dx$$

**Exercise 3.** Let $\{f_n\} \subseteq R(I; \mathbb{C})$ be integrable on $I = [0, 1]$ and suppose that $f_n \to f$ uniformly on $[0, 1]$. Show that, if $\{b_n\} \subseteq [0, 1]$ is an increasing sequence with $\lim_{n\to\infty} b_n = 1$, then

$$\lim_{n\to\infty} \int_{0}^{b_n} f_n(x)\, dx = \int_{0}^{1} f(x)\, dx.$$

**Exercise 4.** Please do the following:

1. Prove that the geometric series
$$\sum_{k=0}^{\infty} x^k$$
converges uniformly to $1/(1-x)$ on any closed interval $[a, b] \subseteq (-1, 1)$.
2. Define
$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} e^{ikx}$$
for $x \in \mathbb{R}$.
   - Prove that the series for $f$ converges uniformly on $\mathbb{R}$ and $f$ is continuous.

- For each $n \in \mathbb{Z}$, compute
$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

**Exercise 5.** Please do the following:

1. Find all nonzero vectors orthogonal to $(1,-1,0)$ which lie in the plane $z = x$.
2. Find all nonzero vectors orthogonal to the vector $(3,2,-5)$ whose components sum to 4.
3. Final an equation of the plane containing the point $(1,0,1)$ with normal $(-1,2,1)$.
4. Adapt the proof of the Cauchy-Schwarz inequality to prove that,
$$|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$$
if and only if $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$, or $\mathbf{x}$ is parallel to $\mathbf{y}$. In other words, for non-zero vectors $\mathbf{x}$ and $\mathbf{y}$, the Cauchy-Schwarz inequality is an equality if and only if $\mathbf{x}$ and $\mathbf{y}$ are parallel.

**Exercise 6.** Please do the following:

1. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$ be non-zero vectors and consider the parallelogram
$$\mathcal{P} = \{(x,y,z) = t\mathbf{a} + s\mathbf{b} \,\big|\, t,s \in [0,1]\}.$$
Prove that the area of $\mathcal{P}$ is $\|\mathbf{a} \times \mathbf{b}\|$.
2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be non-zero and consider the parallelepiped
$$\mathcal{P} = \{(x,y,z) = t\mathbf{a} + s\mathbf{b} + r\mathbf{c} \,\big|\, t,s,r \in [0,1]\}$$
Prove that the volume of $\mathcal{P}$ is $|\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}|$.

*Note: The area of a parallelogram with base $b$ and altitude $h$ is given by $bh$ and the volume of a parallelepiped is the area of the base times its altitude.*