Math 165: Homework 2

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, February 27th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Given the length of the following exercise, it will be worth double a normal exercise. So, this homework has 5 exercises in total that you should turn in. Still, I have included a sixth exercise that can be done for extra credit. The exercise is one of my favorite applications involving integration, improper integration, the monotone convergence theorem. I promise it is fun!

Exercise 1. In class, we used the fact that

$$e^x e^y = e^{x+y}$$

We shall give a proof through the following steps. Please do all of them and, as usual, justify your reasoning.

1. We know that, for any number z, the following series converges absolutely and defines the number e^z:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{z^k}{k!}.$$

   In fact, you can take for granted that all of this works when z is complex. It's amazing! In particular, if x and y are any two complex numbers,

$$e^{x+y} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(x+y)^k}{k!}$$

   Use the binomial theorem to write the approximants in the form

$$\sum_{k=0}^{n} \frac{(x+y)^k}{k!} = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{1}{j!}\frac{1}{(k-j)!}x^j y^{k-j}$$

   for each $n \in \mathbb{N}$.

2. Now, I want you to switch the order of summation to show that

$$\sum_{k=0}^{n}\sum_{j=0}^{k} \frac{1}{j!}\frac{1}{(k-j)!}x^j y^{k-j} = \sum_{j=0}^{n}\sum_{k=j}^{n} \frac{1}{j!}\frac{1}{(k-j)!}x^j y^{k-j}$$

   for each $n \in \mathbb{N}$. *Note: This is not completely obvious. Explain why the bounds on these double sums are as I claimed.*

3. Using what you found above, you can change the index of the inner summation (by setting $l = k - j$) and simplify as much as possible to show that, for $n \in \mathbb{N}_+$,

$$\sum_{k=0}^{n} \frac{(x+y)^k}{k!} = \sum_{j=0}^{n} \frac{x^j}{j!} \sum_{l=0}^{n-j} \frac{y^l}{l!}$$

4. To show that $e^{x+y} = e^x e^y$, we must now take a limit as $n \to \infty$. Naively, it seems that we can just set $n = \infty$ and we'd be done. However, this isn't so straightforward because the inner summation depends on j (in the upper summation index) and this makes things delicate enough that we have to worry a little bit. As it turns out, there are several ways to compute this limit and get what we want. Here, I'll show you a very powerful way which makes use of a famous theorem called *Lebesgue's dominated convergence theorem* or the *LDCT*.

While you won't see a proof of this theorem until MA439, it's actually quite straightforward to prove and so we'll take it for granted. To this end, write

$$\sum_{j=0}^{n}\frac{x^j}{j!}\sum_{l=0}^{n-j}\frac{y^l}{l!}=\sum_{j=0}^{\infty}f_n(j)$$

where

$$f_n(j)=\frac{x^j}{j!}\chi_{[0,n]}(j)\sum_{l=0}^{n-j}\frac{y^l}{l!}$$

and $\chi_{[0,n]}$ is the characteristic function of the set $[0,n]$ which is 1 exactly when $j=0,1,2,\dots,n$ and 0 for $j>n$. Show that, for each $j=0,1,\dots,$

$$\lim_{n\to\infty}f_n(j)=\frac{x^j}{j!}\lim_{n\to\infty}\chi_{[0,n]}(j)\sum_{l=0}^{(n-j)}\frac{y^l}{l!}=\frac{x^j}{j!}e^y=:f(j).$$

Show also that

$$|f_n(j)|\leq\frac{|x|^j}{j!}e^{|y|}:=g(j)$$

for every n.

5. Now, Lebesgue's dominated convergence theorem says that:

$$\text{If}\qquad\sum_{j=0}^{\infty}g(j)<\infty,\qquad\text{then}\qquad\lim_{n\to\infty}\sum_{j=0}^{\infty}f_n(j)=\sum_{j=0}^{\infty}\lim_{n\to\infty}f_n(j)=\sum_{j=0}^{\infty}f(j).$$

Please verify that $\sum_{j=0}^{\infty}g(j)<\infty$.

6. Since the hypothesis of the LDCT is met, use the theorem and put everything together to conclude that

$$e^{x+y}=e^xe^y.$$

For the remainder of the homework, you will work through some basic calculus for complex functions of a real variable. As we discussed in class, this calculus is only ϵ more difficult than what we have already done. I really see the remainder of this homework as simply an opportunity to get you comfortable with complex-valued functions. In what follows, we shall take I to be an interval of the form $[a,b]$, (a,b), $[a,b)$ or $(a,b]$. A complex-valued function on I is a function $f:I\to\mathbb{C}$ of the form

$$f(x)=u(x)+iv(x)$$

where u and v are real-valued functions on I. We call u and v the real and imaginary parts of f, respectively, and sometimes write $u=\mathrm{Re}(f)$ and $v=\mathrm{Im}(f)$. We say that a complex-valued function f is real-valued if $v=\mathrm{Im}(f)$ is identically 0 on I. Below, we extend some of the calculus we've done so far in honors calculus to this complex-valued setting. You should verify, at all stages, that our theory here recaptures the real-valued theory when the complex-valued functions have everywhere vanishing imaginary parts.

1 Differential Calculus

We shall focus mostly on those complex-valued functions $f:I\to\mathbb{C}$ which are continuous, i.e., complex-valued functions $f=u+iv$ whose real an imaginary parts u and v are continuous on the interval I. The set of such functions is denoted by $C^0(I)$ and explicitly defined by

$$C^0(I)=\{f:I\to\mathbb{C}:u=\mathrm{Re}(f)\text{ and }v=\mathrm{Im}(f)\text{ are continuous real-valued functions on }I\}$$

A function $f = u + iv$ is said to be differentiable on I provided that

$$u'(x) = \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \quad \text{and } v'(x) = \lim_{h\to 0} \frac{v(x+h)-v(x)}{h}$$

exist at each[1] $x \in I$. If we know that f is differentiable on I, we define its derivative f' by

$$\frac{df}{dx} = f'(x) = u'(x) + iv'(x)$$

for $x \in [0,1]$. When it happens that f' it itself a continuous function on I, i.e., $f' \in C^0(I)$, we say that f is continuously differentiable[2] on I and we denote the set of all such functions by $C^1(I)$, i.e.,

$$C^1(I) = \{f \in C^0(I) : f \text{ is differentiable on } I \text{ and } f' \in C^0(I)\}.$$

Continuing inductively, for each $k \geq 2$, we say that f is k-times differentiable on I provided that the (k-1)th derivative of f is itself differentiable on I and we write

$$\frac{d^k f}{dx^k} = f^{(k)}(x) = \frac{d}{dx} f^{(k-1)}(x) = \frac{d^k}{dx^k} u(x) + \frac{d^k}{dx^k} v(x)$$

for $x \in I$. The set of k-times continuously differentiable functions is defined by

$$C^k(I) = \{f \in C^{k-1}(I) : f^{(k)} \text{ exists and is a member of } C^0(I)\}.$$

With these sets at hand, we say that a function f is smooth or infinitely differentiable on I if it belongs to $C^k(I)$ for all k, i.e.,

$$f \in C^\infty(I) := \bigcap_{k=0}^{\infty} C^k(I).$$

We have the following proposition:

Proposition A. *Let f and g be complex-valued functions on I and let α and β be complex numbers.*

1. *If f and g are differentiable on I, then so are $\alpha f + \beta g$, fg and f/g (provided that $g \neq 0$ on I) and we have the formulas*
   - *(a)* $\frac{d}{dx}(\alpha f + \beta g)(x) = \alpha f'(x) + \beta g'(x)$ *for* $x \in I$.
   - *(b)* $\frac{d}{dx}(fg)(x) = f'(x)g(x) + f(x)g'(x)$ *for* $x \in I$.
   - *(c)* $\frac{d}{dx}(f/g)(x) = (f'(x)g(x) - f(x)g'(x))/g(x)^2$ *for* $x \in I$.
2. *For any $k = 0, 1, 2 \ldots \infty$, if $f, g \in C^k(I)$, then $\alpha f + \beta g \in C^k(I)$. In particular, this guarantees that each $C^k(I)$ is a subspace of the vector space of complex-valued functions on I and is therefore a vector space in its own right.*

Though the above proposition might seem obvious (at least the first item) and it does follow from the results of single-variable (real-valued) calculus, you should think about what is meant by all of the algebraic operations above. As a good check, here is a basic exercise:

Exercise 2. Writing $f(x) = u(x) + iv(x)$ and $\alpha = a + ib$. Verify that

$$\frac{d}{dx}(\alpha f)(x) = \alpha f'(x)$$

(whenever f is differentiable) by expanding both sides using complex multiplication and applying the definition of the derivative of a complex-valued function given above.

[1] If either endpoint a or b is included in I (e.g, $I = [a,b]$), these limits are taken to be one-sided limits when necessary (either $h > 0$ at $x = a$ or $h < 0$ at $x = b$).

[2] Which is, admittedly, a terrible name.

Exercise 3 (The complex exponential function)**.** In class, we proved (while making use of the results of Exercise 1) that

$$e^{ix} = \cos(x) + i\sin(x) \tag{1}$$

whenever $x \in \mathbb{R}$. In this way, $x \mapsto e^{ix}$ is a complex-valued function on $I = \mathbb{R}$.

1. While we know from Exercise 1 that $e^{i(x+y)} = e^{ix+iy} = e^{ix}e^{iy}$, I want you to use the right hand side of (1) to confirm this. In other words, use known trigonometric identities and the algebra of complex numbers to show that
$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y) = (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) = e^{ix}e^{iy}$$
whenever $x, y \in \mathbb{R}$.

2. Use the above to verify the so-called identity of De Moivre:
$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$$
for $x \in \mathbb{R}$. This identity is key to understanding the nth roots of unity.

3. Upon recalling that, for a non-zero complex number $z = a + ib$, $w = 1/z$ is the unique complex number for which $wz = zw = 1 + 0i$, show that
$$e^{-ix} = \frac{1}{e^{ix}}$$
for each $x \in \mathbb{R}$ (note, by e^{-ix}, I mean $e^{i(-x)}$).

4. Show that e^{ix} is differentiable on $\mathbb{R}$ and
$$\frac{d}{dx}e^{ix} = ie^{ix}$$
and conclude that $e^{ix} \in C^1(\mathbb{R})$. You can use either the definition via power series (and that we can differentiate power series, something I stated but haven't yet proved) or you can use your knowledge of trigonometric derivatives.

5. Using mathematical induction, show that e^{ix} is k-times differentiable on $\mathbb{R}$ with
$$\frac{d^k}{dx^k}e^{ix} = (i)^k e^{ix}$$
for every $k = 0, 1, 2, \dots$. Conclude that $f \in C^k(\mathbb{R})$ for every k and so $x \mapsto e^{ix}$ is smooth.

2 Integral Calculus

In this subsection, we focus on the case in which $I = [a, b]$ where $a < b$ are both finite.

Definition B. Let $f : I \to \mathbb{R}$ be given by $f = u + iv$. If $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ are Riemann/Darboux integrable, i.e., $u, v \in R(I)$, we say that f is Riemann/Darboux integrable on I, write $f \in R(I; \mathbb{C})$ and define the integral of f on I to be the complex number

$$\int_a^b f = \int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx.$$

Proposition C (The integral is linear)**.** *Given an interval I, let $f, g \in R(I; \mathbb{C}$ and α and β be complex numbers. Then $\alpha f + \beta g \in R(I; \mathbb{C})$ and*

$$\int_a^b (\alpha f + \beta g) = \alpha\int_a^b f + \beta\int_a^b g.$$

Exercise 4. As with the analogous proposition for differentiation, the proposition above seems obvious. Still, to get a feel for what it is saying, please prove the following special case (using only results of single-variable calculus): For $f = u + iv \in R(I;\mathbb{C})$ and $\alpha = a + ib \in \mathbb{C}$, show that $\alpha f \in R(I;\mathbb{C})$ and

$$\int_a^b \alpha f = \alpha \int_a^b f.$$

Exercise 5 (The fundamental theorem of calculus and integration by parts)**.** Fix $I = [a, b]$. For a function $f = u + iv \in C^0(I)$, we say that $F = U + iV \in C^1(I)$ is an antiderivative for f on I if $F' = f$ on I.

1. If F is an antiderivative of f on I, use the single-variable calculus version of the fundamental theorem of calculus to show that
$$\int_I f = F(x)\big|_a^b = F(b) - F(a).$$
2. Use the above FTC for complex-valued function to prove the following complex-valued intergration by parts formula: For any $f, g \in C^1(I)$,
$$\int f'g\,dx = f(x)g(x)\Big|_a^b - \int_I fg'.$$

3 A fun optional exercise putting together ideas of convergence and improper Riemann integration

Exercise 6. Let f be continuous, increasing and concave down on the interval $[1, \infty)$ as shown in Figure 1.

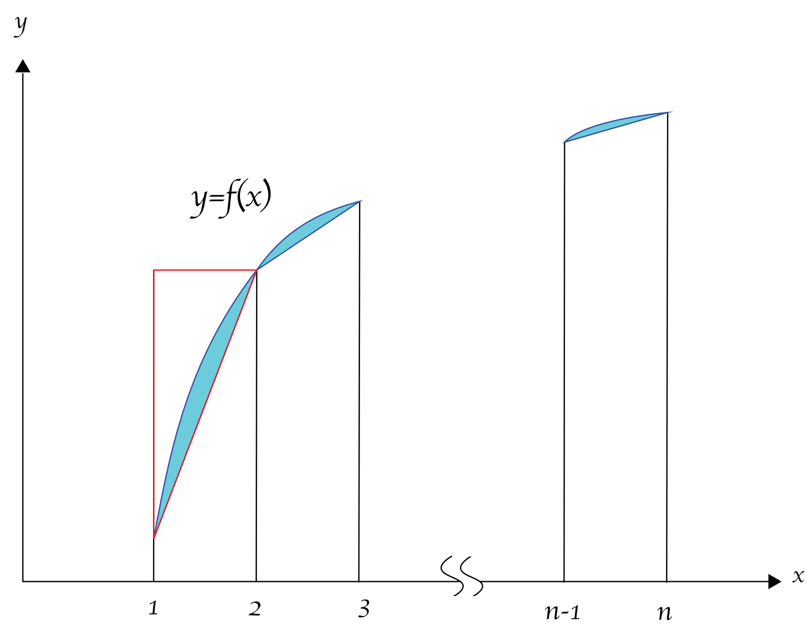

Figure 1: A continuous function f which is increasing and concave down.

For each natural number n, we let A_n be the total area of the regions shown in blue in the figure. This area A_n is formed by taking the area under the graph of f and subtracting the area of the trapezoids (drawn in the figure) beneath the graph from 1 to n. For example, noting that the area of a trapezoid is the base times average of the side lengths (heights), we have

$$\begin{aligned} A_3 &= \int_1^3 f(x)\,dx - 1\cdot\frac{f(1)+f(2)}{2} - 1\cdot\frac{f(2)+f(3)}{2} \\ &= \int_1^2 f(x)\,dx - \frac{f(1)+f(2)}{2} + \int_2^3 f(x)\,dx - \frac{f(2)+f(3)}{2} \\ &= \sum_{k=1}^{2}\left(\int_k^{k+1} f(x)\,dx - \frac{f(k)+f(k+1)}{2}\right). \end{aligned}$$

1. State the general formula for the sequence A_n.

2. Our assumption that f is concave down (look at Figure 1) means that the secant line between any two points on the graph of f lies (except for its endpoints) entirely beneath the graph of f. As a consequence, for each $k = 1, 2, \ldots, n$,
$$(1-t)f(k) + tf(k+1) < f(t+k) \tag{2}$$
for all $0 < t < 1$. Use properties of the integral (monotonicity of the integral[3]) to show that, for each $k = 1, 2, \ldots, n$,
$$\frac{f(k)+f(k+1)}{2} < \int_k^{k+1} f(x)\,dx.$$
Hint: Integrate the inequality (2) with respect to t from $t = 0$ to $t = 1$; the integer k should be treated as a constant. For your integral on the right hand side, make the change of variables $x = t + k$.

3. Use your results of the previous two parts to conclude that the terms (summands) of A_n are positive. Explain why this guarantees that the $\{A_n\}$ is an increasing sequence.

4. By a careful study of the geometry in Figure 1, it can be shown that, for each $n = 1, 2, 3\ldots$, A_n is bounded above by T where T is the area of the red triangle in the figure. In other words,
$$A_n \le T = \frac{f(2)-f(1)}{2} \tag{3}$$
for each $n = 1, 2, 3, \ldots$. Use Figure 2 below to explain why (3) is true, in your own words. You will need to explain the following:
   - why each blue sliver can be moved into the red triangle (*Hint: use the concavity property to explain why each secant line may be moved into the red triangle so that the secant line will not intersect the graph of $f(x)$*)
   - why no two blue slivers will intersect each other in the red triangle, except at the their common endpoint.
   - why no blue sliver will intersect the top of the red triangle. (*Hint: use the increasing and concavity assumption.*)

   You may find it useful to think about moving the second blue sliver into the red triangle and how this construction extends to the remaining blue slivers.

5. Use the monotone convergence theorem to conclude that $\{A_n\}$ converges to some limit $K = \lim_{n\to\infty} A_n$.

6. Now, we focus on a special case. Let $f(x) = \ln x$ for $x \ge 1$. Using the fundamental theorem of calculus, verify that this is continuous, increasing and concave down on $[1, \infty)$.

[3]Here we are using a sort of extra nice version of monotonicity: If $f(x) < g(x)$ for all $x \in (a, b)$, then $\int_a^b f < \int_a^b g$. Though you don't need to prove this, it is easy to do for continuous functions by the ideas of the previous homework.

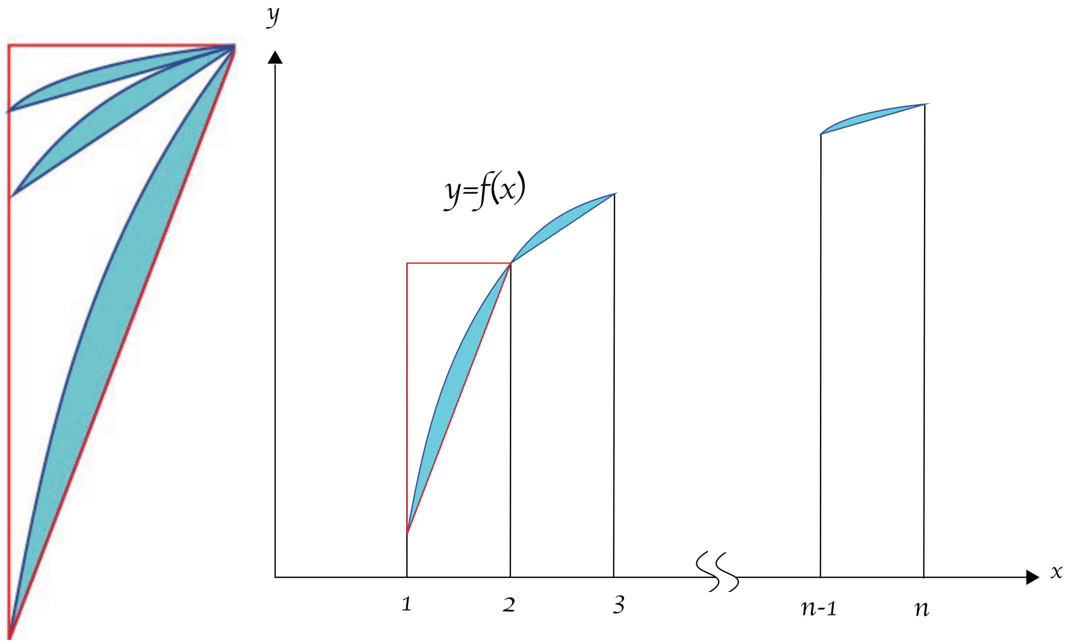

Figure 2: Sliding Slivers

7. Prove that

$$\begin{aligned} A_n &= \int_1^n \ln x\,dx - \left(\frac{\ln 1}{2} + \ln 2 + \cdots + \ln(n-1) + \frac{\ln n}{2}\right) \\ &= n\ln n - n + 1 - \ln n! + \ln\sqrt{n} \\ &= 1 + \ln\left(\frac{(n/e)^n\sqrt{n}}{n!}\right) \end{aligned}$$

and use it and what you have down above to show that

$$\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{n}} = e^{1-K}.$$

In fact, it can be shown that $\lim_{n\to\infty} A_n = K = 1 - \ln(\sqrt{2\pi})$, and so $\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{n}} = \sqrt{2\pi}$. You don't need to determine this constant (what we've done in this exercise isn't enough to do it).

8. From your result above, this means that

$$n! \approx \sqrt{2\pi n}(n/e)^n$$

for large n; this is called Stirling's Formula and it has many applications in mathematics, physics and computer science.

a. Use the formula to approximate 15! and check with your calculator that it's a decent approximation.

b. The number of digits of an integer x is

$$\log_{10}(x) + 1 = \frac{\ln(x)}{\ln(10)} + 1.$$

Use a calculator (e.g. Google or Wolfram Alpha) and Stirling's Formula to approximate the number of digits of 100!. To provide an appreciation of how large 100! is, the (conjectured) number of particles in the known universe is a number with ≈ 87 digits. Using Wolfram Alpha, you can calculate the number of digits of 100! exactly. How close is your approximation from the actual answer?

c. Suppose you flip a fair coin $2n$ times. In an introductory probability/discrete math course (e.g. MA381), you would be able to show that the probability that exactly n heads appear is

$$\frac{(2n)!}{n!n!}\frac{1}{2^{2n}}$$

Applying Stirling's formula, for large n, this is approximately $\frac{1}{\sqrt{\pi n}}$.