Math 165: Homework 1

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus[1]. Your solutions are due on Thursday, February 20th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. Throughout this exercise, $I = [-1, 1]$.

1. For the function $f(x) = x$ and the partition $P = \{-1, -1/2, 1/4, 3/4, 1\}$, compute $\mathcal{U}(f, P)$ and $\mathcal{L}(f, P)$ and draw the pictures that these "areas" represent.
2. If we think of the Riemann/Darboux integral as giving the signed area under the curve, it is reasonable to think that $\int_{-1}^{1} f(x)\,dx = 0$ for the function $f(x) = x$. Can you find a partition P so that $\mathcal{U}(f, P) = \mathcal{L}(f, P) = 0$, or can this zero only be gotten in a limit/supremum/infimum? Explain.

Exercise 2. Consider the interval $I = [0, 1]$ and the collection of regular partitions

$$P_n = \left\{ \frac{j}{n} : j = 0, 1, \dots, n \right\}$$

given for each $n \in \mathbb{N}_+$. For the following bounded and real valued functions on I, compute the lower and upper Darboux sums, $\mathcal{L}(f, P_n)$ and $\mathcal{U}(f, P_n)$, respectively.

1. $$f(x) = \begin{cases} 1 & 0 \le x < 1/4 \text{ and } \frac{3}{4} < x \le 1 \\ 4 & \frac{1}{4} \le x \le \frac{3}{4}. \end{cases}$$
2. $$f(x) = x^2$$
   For this, the following summation formula is sometimes useful $\sum_{j=0}^{n} j^2 = n(n+1)(2n+1)/6$.
3. The Dyadic Dirichlet function:
   $$f(x) = \begin{cases} 1 & x = \frac{k}{2^n} \text{ for some } k, n \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$
   We note that this is different from the Dirichlet function in that, for example, it assigns the value 0 to the fraction 1/3.

Exercise 3. In what follows, f is a bounded real value function on an interval $I = [a, b]$. Prove (or refute with a counterexample) the following statements:

1. If there exists a sequence of partitions $\{P_n\}$ for which
   $$\limsup_n \left(\mathcal{U}(f, P_n) - \mathcal{L}(f, P_n) \right) = 0,$$
   then f is Riemann/Darboux integrable.
2. If there exists a sequence of partitions $\{P_n\}$ for which
   $$\liminf_n \left(\mathcal{U}(f, P_n) - \mathcal{L}(f, P_n) \right) = 0,$$
   then f is Riemann/Darboux integrable.

[1] Now is an excellent time, if you haven't already, to read the syllabus carefully.

3. In which of the previous cases must it be true that the following limits exists and satisfy

$$\lim_{n\to\infty} \mathcal{U}(f, P_n) = \lim_{n\to\infty} \mathcal{L}(f, P_n) = \int_a^b f(x)\,dx?$$

Make sure to say precisely what you are assuming and give a proof – or provide a counterexample.

Exercise 4. In this exercise, let f be a bounded real-valued function on an interval $I = [a, b]$.

1. Use the results of the previous exercise to prove the statement: If there exists a sequence of partitions $\{P_n\}$ for which
$$\lim_{n\to\infty} (\mathcal{U}(f, P_n) - \mathcal{L}(f, P_n)) = 0,$$
then f is Riemann/Darboux integrable and the following limits both hold (in particular, they exist):
$$\lim_{n\to\infty} \mathcal{U}(f, P_n) = \lim_{n\to\infty} \mathcal{L}(f, P_n) = \int_a^b f(x)\,dx.$$

2. If $\lim_{n\to\infty} \mathcal{U}(f, P_n)$ or $\lim_{n\to\infty} \mathcal{L}(f, P_n)$ does not exist, or both limits exist yet they are not equal, can you conclude that f is not Riemann/Darboux integrable? Explain.

3. Use the above to determine which of the functions f in Exercise 2 is Riemann/Darboux integrable, if possible. If it is Riemann integrable, use the previous item to compute the integrals $\int_0^1 f(x)\,dx$.

Exercise 5. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be continuous on I. Prove the following:

1. If, for some $x_0 \in I$, $f(x_0) \neq 0$, then there is some number $\delta > 0$ for which
$$0 < \delta \leq \mathcal{L}(|f|) = \sup_P \mathcal{L}(|f(x)|, P)$$
where the supremum is taken over all partitions P of I. Note, this is the lower Riemann/Darboux integral of the absolute value of f.

2. $\int_a^b |f(x)|\,dx = 0$ if and only if $f(x) = 0$ for all $x \in I$.

3. Does the previous result hold if absolute values are dropped? Does it hold if f isn't continuous?

4. Again, assume that f is continuous on I. Prove that
$$\int_a^c f(x)\,dx = 0$$
for all $c \in I$ if and only if $f(x) = 0$ for all $x \in I$.

Exercise 6. Let $I = [a, b]$.

1. Let $\{g_n(x)\}$ be a sequence of non-negative functions on I and suppose that
$$\lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0.$$
Prove: If f is Riemann/Darboux integrable on I, then
$$\lim_{n\to\infty} \int_a^b g_n(x) f(x)\,dx = 0.$$

2. Prove that, for any Riemann/Darboux integrable function $f(x)$ on $[0,1]$,

$$\lim_{n\to\infty}\int_0^1 x^n f(x)\,dx = 0.$$

Exercise 7. Let f be continuous on $I=[a,b]$ and set

$$M = \sup_{x\in I}|f(x)|.$$

1. Prove that, if $M>0$, then for every $\epsilon>0$, there is some nondegenerate subinterval $J\subseteq I$ for which

$$(M-\epsilon)^n|J| \le \int_a^b |f(x)|^n\,dx \le M^n(b-a)$$

   for all $n\in\mathbb{N}_+$; here $|J|$ means the length of the interval J.

2. Use the above to prove that

$$\lim_{n\to\infty}\left(\int_a^b |f(x)|^n\,dx\right)^{1/n} = M.$$

Exercise 8. Compute the following (and justify your reasoning):

1. $$\int_{-3}^{3}\left|x^2+x-2\right|dx$$

2. $$\int_1^4 \frac{\sqrt{x}-1}{\sqrt{x}}\,dx$$

3. $$\int_0^{\pi/2} e^x\sin(x)\,dx$$

4. $$\frac{d}{dx}\int_1^{x^2} f(t)\,dt$$

   when $f:[0,\infty)\to\mathbb{R}$ is continuous.

5. $$\frac{d}{dt}\int_0^t g(x-t)\,dx$$

   where $g:\mathbb{R}\to\mathbb{R}$ is continuous.

Exercise 9. Use the "First Mean Value Theorem for Integrals" in Wade's book to prove our old favorite version of the mean value theorem for derivatives: If f is once continuously differentiable on $[a,b]$, i.e., $f\in C^1([a,b])$, then there is a $a<c<b$ for which

$$f(b)-f(a)-f'(c)(b-a).$$

Exercise 10. Let $0\le x\le\pi/2$.

1. Use $0\le\cos(x)\le 1$ and the comparison theorem for integrals to prove that $0\le\sin x\le x$.

2. For each non-negative integer m, set

$$s_m(x) = \sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

and

$$c_m(x) = \sum_{k=0}^{m} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Prove

$$s_{2n+1}(x) \leq \sin(x) \leq s_{2n}(x), \qquad s_{2n+1}(x) \leq \sin(x) \leq s_{2n+2}(x)$$
$$c_{2n+1}(x) \leq \cos(x) \leq c_{2n}(x) \quad \text{and} \quad c_{2n+1}(x) \leq \cos(x) \leq c_{2n+2}(x)$$

hold for $n = 0, 1, 2, \ldots,$.

3. Conclude that, for $0 \leq x \leq \pi/2$, $\lim_{m\to\infty} s_m(x)$ and $\lim_{m\to\infty} c_m(x)$ both converge and give the sine and cosine of x, respectively.

Finally, note that Exercises 4 and 5 from Section 5.3 (in the second edition of Wade) on the natural log and the exponential functions are fabulous exercises and you should feel free to try them! (Though you need not turn them in).