Exercise 1. Let f be continuous, increasing and concave down on the interval $[1, \infty)$ as shown in Figure 1.

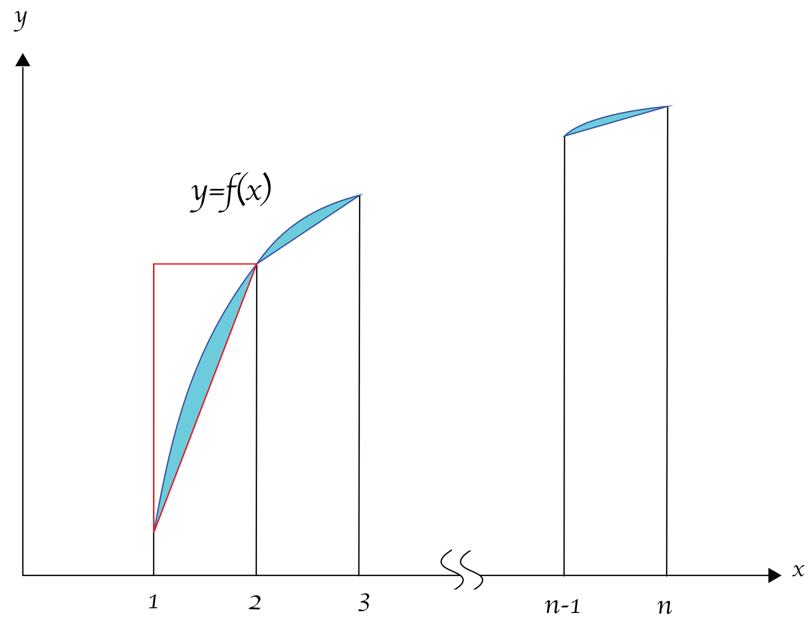

Figure 1: A continuous function f which is increasing and concave down.

For each natural number n, we let A_n be the total area of the regions shown in blue in the figure. This area A_n is formed by taking the area under the graph of f and subtracting the area of the trapezoids (drawn in the figure) beneath the graph from 1 to n. For example, noting that the area of a trapezoid is the base times average of the side lengths (heights), we have

$$\begin{aligned} A_3 &= \int_1^3 f(x)\,dx - 1\cdot\frac{f(1)+f(2)}{2} - 1\cdot\frac{f(2)+f(3)}{2} \\ &= \int_1^2 f(x)\,dx - \frac{f(1)+f(2)}{2} + \int_2^3 f(x)\,dx - \frac{f(2)+f(3)}{2} \\ &= \sum_{k=1}^{2}\left(\int_k^{k+1} f(x)\,dx - \frac{f(k)+f(k+1)}{2}\right). \end{aligned}$$

a. State the general formula for the sequence A_n.

b. Our assumption that f is concave down (look at Figure 1) means that the line between any two points on the graph of f lives (except for its endpoints) entirely beneath the graph of f. As a consequence, for each $k = 1, 2, \ldots, n$,

$$(1-t)f(k) + tf(k+1) < f(t+k) \tag{1}$$

for all $0 < t < 1$. Use properties of the integral (monotonicity of the integral[1]) to show that, for each $k = 1, 2, \ldots, n$,

$$\frac{f(k)+f(k+1)}{2} < \int_k^{k+1} f(x)\,dx.$$

[1] That is, if $f(x) < g(x)$ for all $a \le x \le b$, then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$.

Hint: Integrate the inequality (1) from 0 to 1. For your integral on the right hand side, make the change of variables $x = t + k$.

c. Use your results of the previous two parts to conclude that the terms (summands) of A_n are positive. Explain why this guarantees that the $\{A_n\}$ is an increasing sequence.

d. Show that, for each $n = 1, 2, 3 \ldots$, A_n is bounded above by T where T is the area of the triangle in Figure 1 in red. That is, show that

$$A_n \leq T = \frac{f(2) - f(1)}{2}$$

for each n. A convincing (and clear) geometric argument will suffice here. Still, there are more analytic arguments that can be made using, for example, the so-called "mean value theorem for integrals". Give an argument that you find VERY convincing. *Hint: For a geometric argument, think about why all "slivers" making up the blue shaded region can be moved and stacked to fit within the red triangle.*

e. Use the Monotonic Sequence Theorem (page 722) to conclude that $\{A_n\}$ converges. You should note that $\lim_{n\to\infty} A_n$ is the sum of a series. What is this series?

f. Now, we focus on a special case. Let $f(x) = \ln x$ for $x \geq 1$. You may take for granted that this is continuous, increasing and concave down on $[1, \infty)$. It is also useful to remember that $x \ln x - x$ is an antiderivative of $\ln x$, a standard application of integration by parts. Show that

$$\begin{aligned} A_n &= \int_1^n \ln x \, dx - (\ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln n) + \frac{1}{2} \ln n \\ &= n \ln n - n + 1 - \ln n! + \ln \sqrt{n} \\ &= 1 + \ln \left(\frac{(n/e)^n \sqrt{n}}{n!} \right). \end{aligned}$$

Use Item 5 to conclude that

$$k = \lim_{n\to\infty} \frac{n!}{(n/e)^n \sqrt{n}}$$

exists. Though you don't need to show this, it is true that $k = \sqrt{2\pi}$.

g. From your result above, this means that

$$n! \approx \sqrt{2\pi n}(n/e)^n$$

for large n; this is called Stirling's Formula and it has many applications in mathematics, physics and computer science. Use the formula to approximate 15! and check with your calculator that it's a decent approximation.