This is the eighth homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on **Thursday, November 20th** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. Do these exercises to practice working with partial derivatives.

a. Let $f(x,y) = y^2x^4e^x + xe^{\sin(y)} + x^ey^\pi - xy\sin(e^y)$. Find $f_{xyxyy}(x,y)$.

b. A function $u(x,t)$ is a solution of the one-dimensional heat equation if

$$u_t = u_{xx}\,.$$

Consider the function

$$u(x,t) = \sin(\alpha x)e^{-\beta t}$$

for some real constants α and β. Determine what condition on α and β is necessary for $u(x,t)$ to be a solution of the one-dimensional heat equation.

c. A function $u(x,y,t)$ is a solution of the two-dimensional wave equation if

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right)$$

for some constant c that represents the speed of propagation of the waves. Verify that

$$u(x,y,t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$$

is a solution of the two-dimensional wave equation, and solve for the propagation speed c.

Exercise 2. Do these exercises to practice directional derivatives.

a. Find the directional derivative of the given function $f(x,y)$ at the given point (a,b) in the direction indicated by the given angle θ with the x-axis.

   (i) $f(x,y) = x^3y^4 + x^4y^3$, $(a,b) = (1,1)$, $\theta = \frac{\pi}{6}$
   (ii) $f(x,y) = ye^{-x}$, $(a,b) = (0,4)$, $\theta = \frac{2\pi}{3}$
   (iii) $f(x,y) = e^x\cos(y)$, $(a,b) = (0,0)$, $\theta = \frac{\pi}{4}$

b. Find the directional derivative of the given function f at the given point P in the direction of the given vector $\mathbf{v}$.

   (i) $f(x,y) = e^x\sin(y)$, $P = \left(0, \frac{\pi}{3}\right)$, $\mathbf{v} = \langle -6, 8\rangle$
   (ii) $f(x,y) = x^4 - x^2y^3$, $P = (2,1)$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$
   (iii) $f(x,y,z) = \ln(3x+6y+9z)$, $P = (1,1,1)$, $\mathbf{v} = 4\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$

c. Find the maximum rate of change of f at the given point P and determine the direction in which it occurs.

   (i) $f(x,y) = 4y\sqrt{x}$, $P = (4,1)$
   (ii) $f(x,y) = \sin(xy)$, $P = (1,0)$

(iii) $f(x,y,z) = \dfrac{x+y}{z}$, $P = (1,1,-1)$

Exercise 3. Do these exercises to practice optimization.

a. For each function below, find all critical points and classify them as local minima, local maxima, or saddles.

(i) $f(x,y) = x^2 + xy + y^2 + y$

(ii) $f(x,y) = (x-y)(1-xy)$

(iii) $f(x,y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$

(iv) $f(x,y) = e^x \cos(y)$

(v) $f(x,y) = (x^2+y^2)e^{y^2-x^2}$

b. Find the absolute minimum and maximum values of f on the given region D.

(i) $f(x,y) = x^2 + y^2 - 2x$, where D is the solid triangle with vertices $(2,0)$, $(0,2)$, and $(0,-2)$

(ii) $f(x,y) = x^2 + y^2 + x^2y + 4$, where D is the square in the xy-plane with $|x| \leq 1$ and $|y| \leq 1$.

(iii) $f(x,y) = 2x^3 + y^4$, where D is the solid disk of radius 1 in the xy-plane.

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on November 20th)

Problem 1. In this problem we will investigate when a function of two variables is differentiable. Recall the definition of differentiable involves a limit, and is more than just the existence of the partial derivatives. The idea of restricting to a line or curve may be useful.

a. Show that the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined below is differentiable at $(0,0)$.

$$g(x,y) = \begin{cases} \frac{x^2y}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For the remainder of the problem we will consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the equations

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

b. Show that f is continuous at $(0,0)$.

c. For each $(x,y) \neq (0,0)$ find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$.

d. Find $f_x(0,0)$ and $f_y(0,0)$. Explain the steps carefully.

e. Show that $f_x(x,y)$ and $f_y(x,y)$ are not continuous at $(0,0)$.

f. Show, by investigating the limit in the definition of differentiability, that f is not differentiable at $(0,0)$.

Problem 2 (Shrinkflation)**.** A toothpaste company sells toothpaste in a cylindrical tube. The executive board of the company decides to ~~rip off customers~~/save money by reducing the volume of toothpaste in each tube while making it appear that the toothpaste tube hasn't changed (much) in size. Let's suppose for simplicity that the toothpaste tube is in the shape of a right circular cylinder of radius r and height h. In this case, the volume of toothpaste contained is

$$V = \pi r^2 h$$

and their cost is proportional to the volume with

$$C(r,h) = \frac{1}{\pi}V = r^2 h.$$

For this problem, do not use a calculator and give your answers using the local linearization $L(x,y)$.

a. Suppose that the original toothpaste tube has radius $r_0 = 2cm$ and height $h_0 = 12cm$. They executive board decides that, for the sake of appearances, a reduction in the radius of the tube by $0.01cm$ followed by increase in the height by $0.1cm$ make it appear that the tube is larger yet the volume is smaller. To asses whether or not they are right, estimate

$$C(1.99, 12.1) = (1.99)^2(12.1)$$

using the local linearization of the function $C(r,h)$ near $(r_0, h_0) = (2, 12)$. Did they reduce their cost? Explain.

b. Suppose that they try to make the same change with their miniature tubes of toothpaste (the travel kind). Currently, these bottles radius of $r_0 = 1cm$ and $h_0 = 4cm$ and they decide to reduce the radius by 0.01 and increase the height by $0.1cm$ (the same differences from the bigger tube). Will they reduce the cost?

c. If they reduce the cost with both, please explain. If there is a discrepancy, please explain.

Problem 3 (An application of the Chain Rule: Bernoulli's Equation in Hydrodynamics)**.** In the scientific field of hydrodynamics, Bernoulli's equation states that the pressure P (measured in Pascals $= N/m^2$) of a moving incompressible viscous fluid travelling at velocity v and height[1] y is such that

$$f(P,y,v) = P + \rho g y + \frac{1}{2}\rho v^2$$

is constant; here, $g \approx 10N/kg$ is the gravitational constant and ρ is the constant density of the fluid; you may take $\rho = 1000\,kg/m^3$ throughout the problem. Let's suppose that we follow a particle in the fluid and assume that, at time $t_0 = 0$, the particle is of height $y(t_0) = y_0 = 10m$ and has $P(t_0) = P_0 = 150,000N/m^2$, and velocity $v(t_0) = v_0 = 0.5m/s$. With this initial set up, please answer the following questions:

a. If the pressure is held steady (i.e., $dP/dt = 0$) and the fluid is flowing downward with $dy/dt = -1m/s$, does the velocity of the particle increase or decrease? Explain.

b. If the velocity of the particle is held constant and $\frac{dy}{dt} = -1m/s$, does the pressure increase or decrease? Explain

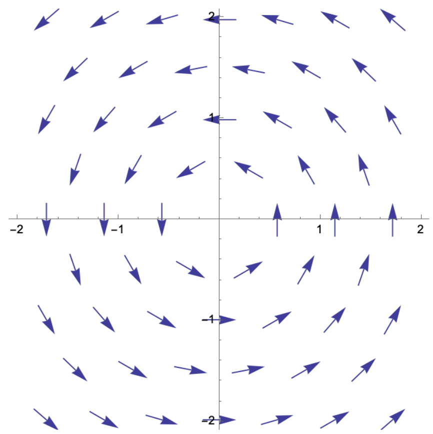
c. Using the chain rule, find a relationship between dP/dt, dy/dt, dv/dt, and v. Explain your formula by explaining various examples, e.g., If I want the pressure to increase/decrease when $v = \ldots$, the velocity and height need to $\ldots$.

d. Suppose that when $h_0 = 10m$, $P_0 = 150,000N/m^2$, $v_0 = 0.5m/s$, you want the pressure to increase the fastest, find the ratio and signs of dy/dt and dv/dt that will do it. Hint: Think in terms of directional derivatives and the gradient.

Problem 4. A *vector field* is a rule that assigns a vector to each point in space. More formally, a vector field F on $\mathbb{R}^2$ is a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ that assigns to each point $(x,y) \in \mathbb{R}^2$ the vector $F(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$; here, F_1 and F_2 are real-valued functions. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is everywhere differentiable, we can think about the gradient $\boldsymbol{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$ as a vector field. In this problem, we study the interesting (and VERY important) question: If you are given a vector field $F(x,y)$, how can we know if/when F is the gradient of a some function f? In other words, is there some $f(x,y)$ for which $F = \boldsymbol{\nabla} f$?

a. Explain why the vectors shown below cannot represent the gradient ∇f of a differentiable function f at different points (even if the function is allowed to be undefined at $(0,0)$). Hint: If the inputs move counterclockwise around a circle centered at the origin, what happens to the values of the function? Why is this a problem?

[1] This can be taken as a height above sea level or any reference point.



We next want to show that there is no differentiable function f with gradient $\nabla f(x, y) = \langle -y, x\rangle$. The gradient of such a function at each point would look similar to the vectors shown in part a (with different lengths). To investigate, we will make use of the function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$,

$$\mathbf{r}(t) = (\cos(t), \sin(t)).$$

The outputs of $\mathbf{r}$ trace out a circle as the input t goes from 0 to 2π.

b. Assuming a differentiable function f exists with $\nabla f(x, y) = \langle -y, x\rangle$, find the derivative of the single variable function $f \circ \mathbf{r} : \mathbb{R} \to \mathbb{R}$.

c. Use the derivative computed in the previous step to compute $(f \circ \mathbf{r})(2\pi) - (f \circ \mathbf{r})(0)$. Explain why this result is nonsense.

d. Repeat steps b and c to show that there is no differentiable function g with gradient

$$\nabla g(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

e. If you recall, Clairaut's theorem says that, for a function $f(x, y)$ which has continuous second-order partial derivatives, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Use this to explain (in a different way) that there cannot exist a function f for which $\nabla f(x, y) = \langle -y, x\rangle$. Does this method work to make the same conclusion about g described in Part d?

Problem 5. Let $f(x, y)$ be a function of two variables which we will assume to be differentiable everywhere. In class, we saw that the gradient $\nabla f(x, y)$ is perpendicular to level curves $f(x, y) = c$. This fact continues to be true in higher dimensions! If $f(x, y, z)$ is a function of three variables, which is differentiable everywhere, then $\nabla f(x, y, z)$ is perpendicular to level surfaces $f(x, y, z) = c$.

a. Consider the surface S defined as the graph $z = f(x, y)$ of the two-variable function $f(x, y) = 9 - x^2 - y^2$. Note that S can also be thought of as the level set $g(x, y, z) = 9$ of the three-variable function $g(x, y, z) = z + x^2 + y^2$.

   (i) Sketch a 2D contour plot of the function f as well as a 3D graph of the surface S.

   (ii) Compute the vectors $\nabla f(1, 1)$ and $\nabla g(1, 1, 7)$ and represent them on your sketches. Which space does each vector live in? What information does each vector tell us about the surface?

   (iii) Find the tangent plane to S at $(1, 1, 7)$ in two ways: first using ∇f and second using ∇g.

b. For each surface S below, find a function $g(x, y, z)$ such that S is a level surface $g(x, y, z) = c$ for some c.

(i) S is the set of all points such that $x = y^2 - z^2$

(ii) S is an infinite cylinder of radius 1 centered around the z-axis

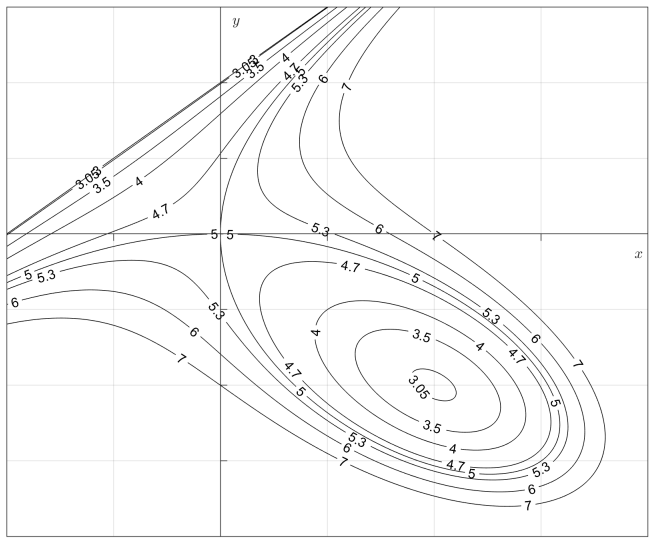
(iii) S is the unit sphere in $\mathbb{R}^3$

(iv) S is the graph of $z = f(x, y)$ for some function $f(x, y)$ of two variables

c. For each of the surfaces in part (a), find an equation for the tangent plane to the surface at a generic point (a, b, c) on that surface.

Problem 6. In this problem, we study level curves, critical points, and make use of the second derivative test.

a. From the level curve plot, determine and classify critical points. Explain your reasoning.



b. Find and classify all critical points of $f(x, y) = 2x^4 - 4xy + y^2 + 4$.

c. Consider the function $g(x, y) = x^2 y - 3$.

(i) Show that $g(x, y)$ has an infinite number of critical points.

(ii) Show that $(0, 0)$ is a critical point where the second derivative test is inconclusive.

(iii) Classify $(0, 0)$ as a local maximum, local minimum, or saddle point by considering the features/geometry of the function.

d. Consider now the function

$$k(x, y) = xy + \frac{2}{x} + \frac{4}{y}$$

defined when $x \neq 0$ and $y \neq 0$.

(i) Find all critical points for k.

(ii) Using the Discriminant/Hessian, classify these critical points as local maxima, minima, or saddle points.

(iii) If any of the points you've found are relative/local extrema, discuss whether or not they are global extrema. Does the function have a global maximum/minimum? Explain.

(iv) Use computational software to plot your function on the grid $-5 \leq x, y \leq 5$. Illustrate your critical point(s).

e. Consider the function

$$q(x, y) = x^4 + y^6$$

(i) Show that q has a single critical point at $(0, 0)$.

(ii) What is the Discriminant/Hessian of q at $(0, 0)$? What does it tell you, if anything, at $(0, 0)$?

(iii) Regardless of your answer for the previous part, give a convincing explanation for why $(0, 0)$ is the unique relative/local and global minimum for the function q.

(iv) Use computational software to plot q on the grid $-2 \leq x, y \leq 2$ and confirm what you argued.