This is the seventh homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on **Thursday, November 6th** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. Please do the following:

a. Find each limit, if it exists, or show that the limit does not exist.

(i) $\displaystyle\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2)$

(ii) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{4-xy}{x^2+3y^2}$

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$

(iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4}$

(v) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

(vi) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4+4y^2}$

b. Determine the set of points at which each function is continuous.

(i) $f(x,y) = \dfrac{xy}{1+e^{x-y}}$

(ii) $g(x,y) = \dfrac{1+x^2+y^2}{1-x^2-y^2}$

Exercise 2. Please do the following:

a. Compute both first partial derivatives of each function below.

(i) $f(x,y) = y^5 - 3xy$

(ii) $f(x,t) = e^{-t}\cos(\pi x)$

(iii) $f(x,y) = (2x+3y)^{10}$

(iv) $f(x,y) = \dfrac{x}{y}$

(v) $f(x,y) = \dfrac{ax+by}{cx+dy}$, where a, b, c, and d are constants.

(vi) $f(x,y) = x^y$

b. Use the limit definition of the partial derivative to find $f_x(x,y)$ and $f_y(x,y)$ for each function below.

(i) $f(x,y) = xy^2 - x^3y$

(ii) $f(x,y) = \dfrac{x}{x+y^2}$

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on November 6th)

Problem 1. Consider the function

$$f(x, y) = \frac{x^2 - y}{x - y}.$$

a. Describe the natural domain of f and the range of f on that domain.

b. Show that f has no limit as (x, y) approaches $(0, 0)$ by computing the limit along two distinct lines through the origin.

c. Sketch the level curves of f corresponding to the values -1, 0, and 1.

d. Explain how your answer to part (c) above also demonstrates that f has no limit as (x, y) approaches $(0, 0)$.

Problem 2. Determine if the following functions are continuous at the given point (a, b). If they are continuous, give a complete argument (that uses ϵs and δs, the squeeze theorem, or something else we've learned that will justify continuity). If the function is not continuous, give an argument that demonstrates it.

a.

$$f(x, y) = \sqrt{x^2 + y^2}, \qquad (a, b) = (0, 0).$$

b.

$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{x^2+y^2}\right) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}, \qquad (a, b) = (0, 0).$$

c.

$$f(x) = \begin{cases} \frac{(x-1)^2 - y^2}{(x-1)^2 + y^2} & (x, y) \neq (1, 0) \\ 0 & (x, y) = (1, 0) \end{cases}, \qquad (a, b) = (1, 0).$$

Problem 3 (Two variable limits are wacky)**.** In this problem, you will demonstrate that a function can have a limit along all (straight) lines through a point and still not have a well-defined limit. Consider the following function:

$$f(x, y) = \frac{2xy^3}{x^2 + y^6}$$

defined for $(x, y) \neq (0, 0)$. Please do the following:

a. Show that, along the line $y = ax$, the function approaches 0 as $(x, y) \to (0, 0)$. Note that, since a is an arbitrary real number, this equation describes an arbitrary non-vertical line.

b. Show that, along the vertical line $x = 0$, the function approaches 0 as $(x, y) \to (0, 0)$. Conclude that, along every line through the origin, $f(x, y)$ approaches 0 as $(x, y) \to (0, 0)$.

c. By considering other curves, show that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ actually does not exist.

Problem 4. For the following functions, compute the partial derivatives by computing the associated limits of difference quotients. That is, for the point (a, b) in question, compute

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

and

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

a. $f(x, y) = x^2 + y$, $(a, b) = (1, 2)$

b. $f(x,y) = x^2 + y$, (a,b), in general.

c. $f(x,y) = \sin(xy)$, $(a,b) = (0,0)$

d.
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0), \end{cases} \qquad (a,b) = (0,0).$$

Problem 5. The **ideal gas law** relates the pressure P, volume V, amount n, and temperature T of a gas via the following formula:

$$PV = nRT$$

(here, R is a constant called the *ideal gas constant*). We can rewrite this to consider the pressure P as a function of three variables:

$$P(n,T,V) = \frac{nRT}{V}.$$

Find expressions for $\frac{\partial P}{\partial n}$, $\frac{\partial P}{\partial V}$, and $\frac{\partial P}{\partial T}$, and give the physical interpretations of these partial derivatives