This is the fourth homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus and hand them in. Your write-ups are due on **Thursday, October 9th** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. Find the radius of convergence and interval of convergence of each series.

a. $\displaystyle\sum_{n=1}^{\infty} n(-x)^n$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-x)^n}{n^2}$

c. $\displaystyle\sum_{n=1}^{\infty} (nx)^n$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

e. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

f. $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

Exercise 2. Find a power series representation for each function and determine the interval of convergence.

a. $f(x) = \dfrac{5}{1-4x^2}$

b. $f(x) = \dfrac{2}{3-x}$

c. $f(x) = \dfrac{1+x}{1-x}$

d. $f(x) = \dfrac{x^2}{a^3-x^3}$

e. $f(x) = \dfrac{x}{(1+4x)^2}$.

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on October 9th)

Problem 1. Fix $a < b$. For the following items, produce a power series that satisfies the following criterion. Also, verify that your power series actually does what you intend (by using tests, etc.).

a. Has domain/interval of convergence (a, b).

b. Has domain/interval of convergence $[a, b]$.

c. Has domain/interval of convergence $(a, b]$.

d. Has domain/interval of convergence $[a, b)$.

Problem 2. We recall that, for $|x| < 1$, we have the series representation

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$$

Use this and the rules about manipulating/differentiating/integrating power series that we learned to show/compute the following:

a. Show that, for $|x| < 1$,

$$\frac{1}{(1-x)^3} = \frac{1}{2}\left(2 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots\right) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n.$$

b. Use the previous item to compute the sum of the series

$$1 - \frac{3}{2} + \frac{6}{4} - \frac{10}{8} + \frac{15}{16} - \frac{21}{32} + \cdots$$

c. Show that, for $|x| < 1$,

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) = \sum_{n=1}^{\infty} \frac{-x^n}{n}$$

d. There is an amazing property of series, due to Niels Henrik Abel, that says the following: Suppose that the power series $f(x) = \sum c_n x^n$ (centered at 0) has radius of convergence $R = 1$; in particular, it converges on the open interval $(-1, 1)$. If this series happens to converge at the left endpoint, i.e., $\sum_n c_n(-1)^n$ converges, then the function $f(x)$ is (right) continuous at $x = -1$ so that, in particular,

$$\lim_{x \to -1} f(x) = \sum_{n=0}^{\infty} c_n(-1)^n.$$

Use this result, called Abel's theorem, to compute the sum of the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}.$$

Problem 3. In this problem, I ask you to produce *closed form* expressions for various series/summations. By closed form, I mean polynomials or rational functions that do not involve any Σ notation or "$\cdots$". For example, we know that the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

can be equivalently represented as $\dfrac{1}{1-x}$ on the interval $(-1, 1)$. In other words, for $|x| < 1$, we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

Use this fact and what you've learned about power series to solve the parts below.

a. Express each series as a rational function (a quotient of two polynomials) on $(-1, 1)$.

(i) $x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \cdots$

(ii) $1 + x^2 + x^4 + x^6 + x^8 + x^{10} + \cdots$

(iii) $x - x^3 + x^5 - x^7 + x^9 - x^{11} + \cdots$

(iv) $5 + 10x + 20x^2 + 40x^3 + 80x^4 + 160x^5 + \cdots$

(v) $2 + x + 2x^2 + x^3 + 2x^4 + x^5 + \cdots$

b. Let n be some natural number. Express each arbitrarily-long finite sum as a closed form rational function.

(i) $x^2 + x^3 + x^4 + \cdots + x^n$

(ii) $1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots \pm x^{2n}$

(iii) $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \cdots + \dfrac{1}{x^n}$

c. Assuming that $|x| < 1$, find a closed form for the infinite sum of series

$$(1 + x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + x^4 + \cdots) + (x^2 + x^3 + x^4 + x^5 + \cdots) + \cdots.$$

Problem 4 (Trigonometric Identities)**.** In this problem, you will use known series expansion of common trigonometric function to verify well-known identities. To this end, recall that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}$$

and

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j+1};$$

these series all converge absolutely on $\mathbb{R} = (-\infty, \infty)$. For the remainder of the problem, please do not use your prior knowledge of these functions. Use only their definition above via power series and the properties you establish.

a. Verify that $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

b. Verify that $\frac{d}{dx} \sin(x) = \cos(x)$ and $\frac{d}{dx} \cos(x) = -\sin(x)$.

c. By considering the function

$$F(x) = \cos^2(x) + \sin^2(x) = (\cos(x))^2 + (\sin(x))^2$$

for $x \in \mathbb{R}$, which is evidently differentiable on $\mathbb{R}$, please do the following

(i) What is $F(0)$?

(ii) Using your normal rules for differentiation and what you learned in Item (b.), compute and simplify the derivative of $F(x)$. *Note: I am not asking you to square any series. Think of the chain rule.*

(iii) By appealing to theorems of single-variable calculus, conclude that

$$\cos^2(x) + \sin^2(x) = 1$$

for all $x \in \mathbb{R}$.

Note: The final identity is the famous *Pythagorean Identity.* While it can be established by many approaches – including squaring series via term-by-term multiplication (akin to "foiling" but more like "foioioioioioioioio$\cdots$"). Te method given here is perhaps the most simple way to establish the identity.