This is the third homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus[1]) and hand them in. Your write-ups are due on **Thursday, October 2nd** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1.

Read Sections 11.5-11.8 of Stewart.

Read Sections 3.8, 3.9 and 4.1 of Gouvêa.

Exercise 2. What is an alternating series? What conditions must an alternating satisfy in order to converge? Is those conditions are satisfied, what can you say about the remainder after n terms?

Exercise 3. Determine whether the following series converge or diverge.

a. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots$

b. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

e. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

f. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$

g. $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-n}$

h. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$

i. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

j. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

[1]Now is a superb time to read the syllabus.

k. $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n})$

Exercise 4. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[3]{n^2}}$$

$$\sum_{n=0}^{\infty}(-1)^{n+1}\cdot\frac{n^2}{n^2+1}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+1}$$

$$\sum_{n=1}^{\infty}\frac{-n}{n^2+1}$$

Exercise 5. Use any test to determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty}\frac{(-1)^n\ln n}{n}$$

$$\sum_{n=1}^{\infty}\frac{(-9)^n}{n10^{n+1}}$$

$$\sum_{n=1}^{\infty}\frac{\sin(n\pi/6)}{1+n\sqrt{n}}$$

Exercise 6. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(kn)!}$$

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on October 2nd)

Problem 1. In each case, either give an example of a series which meets the following criteria, or explain why no such example can exist. If you use any theorems, results, or tests, be sure to mention them.

a) A series $\sum a_k$ such that $\sum a_k$ converges, but $\sum |a_k|$ diverges.

b) A series $\sum a_k$ such that $\sum_{k=1}^{\infty}|a_k|$ converges, but $\sum_{k=1}^{\infty}a_k$ diverges.

c) A series $\sum a_k$ such that $\sum a_k$ and $\sum |a_k|$ both converge.

d) A series $\sum a_k$ such that $\sum a_k$ and $\sum |a_k|$ both diverge.

e) A series which is both conditionally convergent and absolutely convergent.

f) A series $\sum a_k$ such that $a_k \geq 0$ for all k and $\sum a_k$ converges conditionally.

Problem 2. At this point, we have seen many different types of series and many different tests that we can use to determine their convergence. For each series below, *try all of the tests* that apply and report the results of each test. The tests that you should run include

- the divergence test,
- the integral test,
- the comparison test (comparing to some sequence b_n that you come up with yourself),
- the alternating series test, and
- the ratio test.

The results of each test might be

- the test says that the series converges,
- the test says that the series converges absolutely,
- the test says that the series diverges,
- the test is inconclusive, or
- the test is not applicable to this type of series.

For example, if our series was the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, we could report that

- the divergence test is inconclusive because $\lim_{n\to\infty} \frac{1}{n} = 0$,
- the integral test says that the series diverges because $\int_1^{\infty} \frac{1}{x}\, dx = \infty$,
- the comparison test says that the series diverges by comparison with $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots$,
- the alternating series test is not applicable, and
- the ratio test is inconclusive since $\lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$.

a) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$

c) $1 + \dfrac{1}{1\cdot 3} + \dfrac{1}{1\cdot 3\cdot 5} + \dfrac{1}{1\cdot 3\cdot 5\cdot 7} + \dots$

d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{100 + 2001\cdot k}$

e) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$

f) $\displaystyle\sum_{m=1}^{\infty} \frac{m}{4^m}$

Problem 3. Each of the series below converges to some value. Find the sum of the series using any of the tools you have available to you.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}3^{n-1}}{2^{2n}(-7)^{n-1}}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(n+2)!}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{f(n)}}$, where $f(n) = n + (-1)^{n+1}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$, where $a_1 = 1$ and $a_n = \displaystyle\sum_{k=1}^{n-1} a_k$ for $n \geq 2$

e. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n + \pi^n}{(\pi e)^n}$

Problem 4. In class, we learned about the amazing ratio test:

Theorem A. *Given a series $\sum a_n$, suppose that the limit*

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exits. Then, we have the following:

1. If $0 \leq r < 1$, the series $\sum a_n$ converges absolutely.

2. If $r > 1$, the series $\sum a_n$ diverges.

3. Otherwise, if $r = 1$, then nothing can be said and more analysis is required.

a. For practice, use the ratio test to determine the (absolute) convergence/divergence of the following three series:

(i)

$$\sum_{n=1}^{\infty} \frac{(-10)^n}{(2n)!}$$

(ii)

$$\sum_{n=0}^{\infty} (-1)^n \frac{n^{100}}{(1 + \frac{1}{100})^n}$$

(iii)

$$\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n^2 4^n}$$

b. In class, we did not give reasoning why the series diverges when $r > 1$. Please give a convincing argument that explains why this is the case. In other words, explain why $\sum a_n$ must diverges when $\lim_{n\to\infty} |a_{n+1}/a_n| > 1$.

Problem 5. Many series have the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n)^n$.

a. Using the Colab notebook linked below, compute the first 50 terms of the series, $a_1, \ldots, a_{50}$, the first 50 values of $|b_n|$, and the first 50 partial sums. Report your findings. Do they appear to converge? You are welcome (and encouraged!) to test any other function you'd like.

$$\text{(i) } b_n = \frac{2n^2+n}{3n^2+1}, \quad \text{(ii) } b_n = \frac{3n^2}{n+1}, \quad \text{(iii) } b_n = \frac{-n}{n+1}, \quad \text{(iv) } b_n = \frac{n+3}{3n^2+1}$$

b. For all of the functions from part (a), compute $\lim_{n\to\infty} |b_n|$.

c. Using parts (a) and (b), what can you say about conditions for convergence of series of the form $\sum_{n=1}^{\infty} (b_n)^n$?

d. You should see from the experiments that the convergence of these types of series depends on $\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} |a_n|^{1/n}$. Suppose that for some large enough N, we can say that $|a_n|^{1/n} < r$, for some value of r and all $n \geq N$. Show that $\sum_{n=N}^{\infty} |a_n|$ can be bounded above by a geometric series.

e. Using the previous part and the fact that

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|,$$

find the values of r for which the series converges. Explain your reasoning.

f. Now, consider the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. To write it in the form of the previous parts of this problem, we have $\sum_{n=1}^{\infty} \left(\frac{n^{2/n}}{2}\right)^n$, which requires computing $\lim_{n\to\infty} \frac{n^{2/n}}{2}$. Use the following two facts to find the limit, demonstrating how the root test can be applied in this case:

(i) $\ln(a^b) = b\ln(a)$ for $a > 0$,

(ii) If $\lim_{n\to\infty} a_n = L$ and f is a continuous function where $f(n) = a_n$, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Colab link: `https://colab.research.google.com/drive/19yZUvr2oSAi68jLfphYAYE71WlcMFBTA?usp=sharing`

Problem 6. Consider the power series

$$\sum_{n=2}^{\infty} \frac{(-2)^n}{n\ln(n)} (x+2)^n.$$

Perform a complete analysis of this power series. In doing so, find:

a. All numbers x for which the series converges absolutely

b. All numbers x for which the series diverges

c. All number x for which the series converges conditionally

d. The interval and radius of convergence

In the course of your analysis, please make sure to identify which test(s) were used to make each conclusion.