This is the second homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus[1]) and hand them in. Your write-ups are due on **Thursday, September 25th** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. Use the integral test to determine whether the series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$.

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Exercise 2. Determine whether the series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

c. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \dots$

d. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \dots$

e. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$

f. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

g. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

Exercise 3. Explain why the integral test can't be used to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$ is convergent.

Exercise 4. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to 3 decimal places.

[1]Now is a superb time to read the syllabus.

Exercise 5. Find the sum of the series $\sum_{n=1}^{\infty}(2n+1)^{-6}$ correct to 5 decimal places.

Exercise 6. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is convergent.

a. If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?

b. If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

Exercise 7. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent.

a. If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?

b. If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

Exercise 8. Determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$

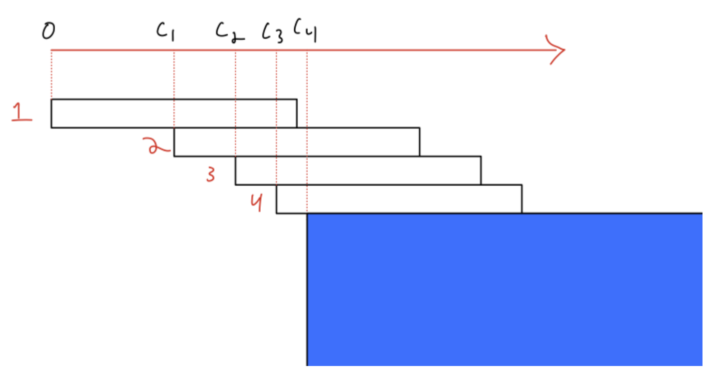
d. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

e. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$

f. $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on September 25th)

Problem 1. I want to stack 4 blocks on the end of table, as shown in the picture, so that they reach as far over the edge of the table as possible without toppling. Accordingly, the center of mass of the top block must be supported by the second block, and the center of mass of the combined top two blocks must be supported by the third block, etc.



If c_k is the center of mass of the stack containing the the top k blocks ($k = 1, 2, 3$ or 4), measured horizontally from the left end of the top block, its must coincide with the left end of block $(k + 1)$ (or the edge of the table if $k = 4$). Assume each block has length 1 foot, and is uniform, so the center of mass of one block is halfway along the block. Then the center of mass of the stack of the first k blocks will be the weighted average of the center of mass of block number k and the center of mass of the combined first $k - 1$ blocks, as follows:

$$c_k = \frac{(k-1)c_{k-1} + (c_{k-1} + 1/2)}{k} = \frac{kc_{k-1} + 1/2}{k}.$$

a) Find the horizontal distance the stack of 4 blocks extends past the edge of the table.

b) For an unknown positive integer n, write down a sum which gives the horizontal distance the stack extends from the table if I build it out of n blocks instead of 4 blocks.

c) If I have an unlimited supply of blocks, patience, vertical space, and precision, is there any limit to how far I can make the stack extend horizontally?

d) How many blocks do I need for the stack to extend one yard (three feet) past the edge of the table? Hint: You can use your results from Problem 5 of HW1.

Problem 2. Suppose that $f(x)$ is a continuous, positive, decreasing function on $[0, \infty)$, and let $a_n = f(n)$. Assume that the integral

$$\int_0^\infty f(x)\,dx$$

converges. From the remainder estimate for the integral test, we know that

$$\int_1^\infty f(x)\,dx \leq \sum_{n=1}^\infty a_n \leq \int_0^\infty f(x)\,dx\,.$$

We can think of the integrals on each side as under- and over-estimates of the actual sum of the series.

a. Explain why it must be true that

$$\sum_{n=1}^\infty a_n = \int_c^\infty f(x)\,dx$$

for some real number c.

b. Find the value of c that satisfies the equation above in part (a) for the specific case $f(x) = e^{-x}$.

c. The idea behind the remainder estimate for the integral test is that we can think of the series in the middle as a left or right "Riemann sum" for the improper integrals on the sides. What if we want to think of the series as a *midpoint* Riemann sum? Under the additional hypothesis that $f''(x) > 0$ for all $x > 0$, draw a picture to explain why it *should* be true that

$$\sum_{n=1}^\infty a_n \leq \int_{\frac{1}{2}}^\infty f(x)\,dx\,. \qquad (1)$$

Two things to note: First, we are working now with a general f and so not necessarily the one from part (b). Second, your argument should make use of the fact that f's graph is concave up (which is what we know $f''(x) > 0$ means).

d. Please follow the steps below in which you will show that, in fact, the inequality (1) holds when f satisfies $f''(x) > 0$.

(i) Let k be some positive integer. Show that

$$\int_k^{k+\frac{1}{2}} f(x)\,dx > \frac{1}{2}f(k) + \frac{1}{8}f'(k)\,,$$

(Hint: Starting with the inequality $f''(t) > 0$, integrate three times: first from $t = k$ to $t = y$, second from $y = k$ to $y = z$, and third from $z = k$ to $z = k + \frac{1}{2}$.)

(ii) As before, let k be some positive integer. Show that

$$\int_{k-\frac{1}{2}}^{k} f(x)\,dx > \frac{1}{2}f(k) - \frac{1}{8}f'(k)\,.$$

(iii) Putting (i) and (ii) together, we get that

$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(x)\,dx > f(k)\,.$$

Use this fact to obtain the desired inequality:

$$\sum_{n=1}^{\infty} a_n \le \int_{\frac{1}{2}}^{\infty} f(x)\,dx\,.$$

e. Returning to the example where $f(x) = e^{-x}$, verify that the inequality (1) does holds for this specific f. You may use your results from (b).

Problem 3. By now we've seen many ways to decide whether a series converges or diverges (and we'll see more ways soon too!). Try to answer the following with what we've learned so far.

a. Does $\displaystyle\sum_{n=1}^{\infty} \frac{3+\pi^n}{e^n}$ converge or diverge? Justify your answer carefully.

b. What about $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+1}$, does it converge or diverge?

c. Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^k}$ converge or diverge? Justify your answer carefully.

d. Suppose $a_k = \frac{2}{k+1}$ for $1 \le k \le 100000$ and $a_k = \frac{\ln k}{k^{1.5}}$ for all $k > 100000$. Decide whether $\displaystyle\sum_{k=1}^{\infty} a_k$ converges or diverges, and justify your answer carefully. (Possible hint: try to show that $a_k < \frac{1}{k^{1.3}}$ for $k > 100000$.

Problem 4. Exploring p-series.

a. We looked at the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$. Modify this slightly to examine the behavior of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,(\ln(n))^p}$.

(i) Let $p = 1$. Plot $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x\ln(x)}$, and $h(x) = \dfrac{1}{x^2}$. What do you observe? Which function decays to $y = 0$ faster? What does this mean for the logarithmic series?

(ii) For $p = 1$, determine whether the logarithmic series converges or diverges.

(iii) Choose a few other values of p and plot $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{x\,(\ln(x))^p}$, and $h(x) = \dfrac{1}{x^p}$. What do you notice?

(iv) Determine the values of p for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\left(\ln(n)\right)^p}$ converges.

b. We know $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$ converges (why?). Find a partial sum that is within 0.00001 of the exact sum. Explain how you know. (Leonhard Euler actually found the exact value of this series to be $\pi^4/90$, but don't use that fact to do the problem!)

c. We also know $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges (why?). Still, we can estimate its partial sums using integration techniques. To within a tolerance of 1, how big is $\displaystyle\sum_{k=1}^{N} \frac{1}{\sqrt{k}}$ for $N = 1000$? $N = 10^6$? $N = 10^{80}$?

Problem 5. Determine whether the following series are absolutely convergent, conditionally convergent or neither.

a)

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{2^k}$$

b)

$$\sum_{k=2}^{\infty} (-1)^k \ln\left(1 + \frac{1}{k}\right)$$

c)

$$\sum_{k=2}^{\infty} (-1)^k \frac{1}{\sqrt{k^2 - 1}}$$

d)

$$\sum_{k=7}^{\infty} \frac{\cos(7k)}{7^k + k^7 + \sin^7(7k)}$$

Problem 6. a. Give a careful statement of the alternating series test.

b. Does the series with alternating terms

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} \dots$$

converge or diverge? Be careful!

c. Consider the alternating harmonic series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

(i) This series converges by the alternating series test. Double check this: verify that the alternating series test works here to show the series converges.

(ii) Can you find a number n such that the nth partial sum S_n is within 0.05 of the sum of the infinite series? And for your n, will S_n be an overestimate or underestimate of this sum?

(iii) Calculate your S_n as a decimal number using a calculator or computer.

(iv) Now we change the order of the terms around, we can form a new series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \dots$$

Explain why the alternating series test *cannot* be used to conclude convergence of this new series.

(v) In fact, this new rearranged series converges. Compute the partial sums S_9, S_{12}, S_{15}, and S_{18} of this series in decimal form, with calculator or a computer. Does this series seem to converge to the same sum as the alternating harmonic series?