This is the zeroth homework assignment for Math 160 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (or the nightly TA sessions). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus[1]) and hand them in. Your write-ups are due on **Thursday, September 11th** in the box outside my office door. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1 (This week's reading)**.** Please do the following:

a. Read Chapter 1 and Sections 2.1-2.3 of A Short Book on Long Sums by Fernando Gouvêa.

b. Read Section 11.1 in Multivariable Calculus by James Stewart.

Exercise 2. Please do the following sub-exercises:

a. Answer the following:

 (i) What is a sequence?

 (ii) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?

 (iii) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$.

b. For the following sequences, please list out the first five terms:

 (i)
$$a_n = \frac{2n}{n^2+1}$$

 (ii)
$$a_n = \frac{(-1)^{n-1}}{5^n}$$

 (iii)
$$a_n = \frac{1}{(n+1)!}$$

 (iv)
$$a_1 = 1, a_{n+1} = 5a_n + 3$$

 (v)
$$a_1 = 2,\ a_{n+1} = \frac{a_n}{1+a_n}$$

c. Find a formula for the general terms of the following sequences.

 (i) $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\}$

 (ii) $\{-3, 2, -\frac{4}{3}, \frac{8}{9}, \frac{-16}{27}, \dots\}$

 (iii) $\{1, 0, -1, 0, 1, 0, -1, 0, 1, \dots,\}$

Exercise 3. For the following sequences, calculate (to four decimal places) the first ten terms of the sequence and then use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

[1]Now is a superb time to read the syllabus.

a.
$$a_n = \frac{3n}{1+6n}$$

b.
$$a_n = 1 + (-1)^n$$

c.
$$a_n = 1 + (0.5)^n$$

Exercise 4. Determine whether the sequence converges or diverges. If it converges, find the limit.

a.
$$a_n = 1 + (0.2)^n$$

b.
$$a_n = \frac{3+5n^2}{n+n^2}$$

c.
$$a_n = e^{1/n}$$

d.
$$a_n = \tan\left(\frac{2n\pi}{1+8n}\right)$$

e.
$$a_n = \frac{n^2}{\sqrt{n^3+4n}}$$

f.
$$a_n = \frac{(-1)^n}{2\sqrt{n}}$$

g.
$$a_n = \cos(n/2)$$

h.
$$\left\{\frac{(2n-1)!}{(2n+1)!}\right\}$$

i.
$$\left\{\frac{e^n + e^{-n}}{e^{2n}-1}\right\}$$

j.
$$\{n^2 e^{-n}\}$$

k.
$$b_n = n\sin(1/n)$$

l.
$$c_n = \left(1+\frac{2}{n}\right)^n$$

Part 2 (Solutions for these problems are due in the appropriate box outside my office door at 11:00AM on September 11th)

Problem 1 (Introductions)**.** With the aim of getting to know all of you, please answer the following questions.

a. What is your preferred name, i.e., what should I call you in class, and what pronouns do you use?

b. What is an interesting fact about you?

c. Why are you taking this course? If your answer is "It's required for my (intended) major", what major are you referring to?

d. Beyond mathematics, what subject is most fascinating to you?

e. What was the last math class you took? How did you like it?

f. What role do you think calculus will play in your life after this class?

g. Is there anything you wish me to know about you as a student of calculus?[2]

h. Do you have any dog[3] allergies (or any fear of dogs)?

Problem 2 (Constant and linear approximations)**.** Let $f(x) = \sqrt[3]{x}$. In this problem you will study approximations for $f(128) = \sqrt[3]{128}$ using constant and linear approximations with center point $a = 125$.

a. Compute $f(125)$ exactly.

b. A first guess for $f(128) = \sqrt[3]{128}$ is to use the degree 0 Taylor polynomial for f centered at $a = 125$:

$$T_0(x) = 125.$$

In other words, we simply pretend that f remains constant, so $f(128) \approx T_0(128) = f(125)$. Why is this a good first guess? (hint: f is a continuous function.)

c. Of course, we can do better: rather than a horizontal line approximation, we can use the *tangent line* or *linear approximation* for f centered at $a = 125$; this is the degree 1 Taylor polynomial $T_1(x)$. Compute $T_1(x)$ and $T_1(128)$ exactly.

d. Using graphing software, plot $T_0(x), T_1(x)$, and $f(x)$ on the same set of axes from $x = 124$ to $x = 129$. What do you observe? Using what you see on the graph, discuss how well $T_0(128), T_1(128)$ do at approximating $f(128)$, and whether these are over- or under-approximations.

e. The *error* of our linear approximation at an input x is the difference $f(x) - T_1(x)$. If we can estimate the size of the error, then we can understand how accurate the approximation $T_1(x)$ is. You'll now show how a rough estimate for the size of the second derivative $f''(x)$ lets us estimate the error of the linear approximation, *without any calculator needed to tell us the true value*

(i) Show that

$$-\frac{2}{9 \cdot 5^5} \leq f''(t) \leq 0$$

for $125 \leq t \leq 128$

(ii) Let x be another number in the interval $[125, 128]$. Using integration with respect to t from $t = 125$ to $t = x$, show that

$$-\frac{2}{9 \cdot 5^5}(x - 125) \leq f'(x) - f'(125) \leq 0$$

[2] e.g. "I cannot see orange, so don't use that marker," or "I have a dreadful fear of trigonometric functions," "I am a visual learner" "I am on the (insert sport here) team," etc.

[3] I ask this because sometimes I bring my dog into office hours and, before I bring her in, I want to make sure that everyone is okay with it.

(iii) Let c be another number in the interval $[125, 128]$. Use integration one more time to from $x = 125$ to $x = c$ to show that

$$-M(x-125)^2 \leq f(x) - T_1(x) \leq 0.$$

where $M > 0$ is a constant that you will explicitly provide in your answer.

(iv) What is the approximate size of the error $f(128) - T_1(128)$? And what is the sign of the error? Is $T_1(128)$ an over-approximation or under-approximation? How many decimal places of accuracy do you expect? Answer all of these using the previous part, do NOT answer by computing $\sqrt[3]{128}$ on your calculator.

(v) The upshot of the work in previous parts is that you have now understood the error $f(x) - T_1(x)$ not only at $x = 128$, but also at any $x = c$ in the interval $[125, 128]$. Pick two values of c in $(125, 128)$ and give estimates of the error $f(c) - T_1(c)$ for each.

Problem 3 (Taylor Polynomial Approximation)**.** This problem asks you to explore several functions and associated Taylor polynomial approximations (centered at a point). For the given functions "f" and center point "a", please do the following:

a. For the given point a, compute the first five derivatives of f at a. If there is an easily discernible pattern, give a general formula for $f^{(n)}(a)$ for the nth derivative of f evaluated at a.

b. Write down the first six Taylor polynomials, $T_0(x)$, $T_1(x)$, $\ldots, T_5(x)$.

c. Using graphing software (Sage, Mathematica, Matlab, etc), plot the function f alongside (at least) the first six Taylor polynomials on an interval of the form $(a-1, a+1)$. Describe (precisely) any observations that you make. For example: Do the Taylor polynomials "seem" to approximate f at the point a? How about near the point a?

(i) $f(x) = x^2$ and $a = 0$

(ii) $f(x) = x^2$ and $a = 1$.

(iii) $f(x) = \cos(x)$ and $a = 0$

(iv) $f(x) = \cos(x)$ and $a = \pi$

(v) $f(x) = \frac{1}{1-x}$ and $a = 0$.

Note: For all of the examples above, there is a fairly easily obtainable formula for $f^{(n)}(a)$ for all n.

Problem 4. In this exercise, we explore two somewhat surprising behaviors of Taylor Polynomials.

a. First, consider the function

$$f(x) = \frac{1}{1-x}$$

and its Taylor polynomials centered at $a = 0$.

(i) If you haven't already, obtain a formula for $f^{(n)}(a)$ for all natural numbers n. If you have obtained the formula from the previous exercise, just write it down here.

(ii) With your formula, you should find that the Taylor polynomials can be written, in general, as

$$T_n(x) = 1 + x + x^2 + \cdots + x^n = \sum_{k=0}^{n} x^k$$

As you observed in the previous problem, these Taylor polynomials describe the function f fairly well for $-1 < x < 1$ (i.e., on the interval $(-1, 1) = (a-1, a+1)$). By considering larger and larger values of n, explain (either analytically or graphically) what is happening for $x \geq 1$ as n increases. For example, for $x > 1$, do the Taylor polynomials approximate f? Do they approximate anything?

(iii) Do the same careful analysis for $x \leq -1$.

b. Consider now the function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and the point $a = 0$.

(i) Plot the function f on the interval $(-1, 1)$.

(ii) Use rules of differentiation (e.g., the chain rule) to compute $f'(x)$ at points $x \neq 0$. Also, compute the derivative of f at 0 by its limit definition:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2} - 0}{h} = \lim_{h \to 0} \frac{1}{h} e^{-1/h^2}$$

Hint: By setting $\theta = 1/h$, it makes sense to compute the above limit by noting that

$$\lim_{h \to 0} \frac{1}{h} e^{-1/h^2} = \lim_{\theta \to \infty} \frac{\theta}{e^{\theta^2}};$$

the latter limit can be easily computed suing L'Hôpital's rule[4].

(iii) Extending the argument above (and you should try), it can be shown that, for each n, there is a polynomial $p_n(\theta)$ for which

$$f^{(n)}(0) = \lim_{\theta \to \infty} \frac{p_n(\theta)}{e^{\theta^2}}.$$

Argue that this limit is always zero and so all of f's derivatives vanish at $a = 0$.

(iv) With what you showed above, give the formula for the nth Taylor polynomial of f at $a = 0$.

(v) Plot f alongside its Taylor polynomials and describe your observations.

Problem 5 (Improper Integrals)**.** At first, when the (Riemann) integral is introduced in Calculus 1, one only considers integrating a function f on a closed and bounded interval $[a, b]$. In this case, the integral of a function f on $[a, b]$ is

$$\int_a^b f(t)\, dt$$

and is defined as a limit of Riemann sums. This initial definition of integration doesn't tell you how to interpret "improper" integrals like

$$\int_0^\infty f(t)\, dt \qquad \text{or} \qquad \int_{-\infty}^\infty f(t)\, dt.$$

Let's remedy this. Let $f : [a, \infty) \to \mathbb{R}$ be a (piecewise) continuous function, i.e., f is a function from the interval $[a, \infty)$ into the set of real numbers $\mathbb{R} = (-\infty, \infty)$ which is continuous at every point. Given any fixed $x > a$, it makes sense to consider the integral

$$\int_a^x f(t)\, dt$$

because f is continuous on $[a, x]$. More precisely, the continuity of f on $[a, x]$ guarantees that f is integrable on $[a, x]$. We note that, to each such $x > 0$,

$$I(x) = \int_a^x f(t)\, dt$$

is a real number and hence I is a real-valued function on (a, ∞). If the limit

$$\lim_{x \to \infty} I(x) = \lim_{x \to \infty} \int_a^x f(t)\, dt$$

[4] If you don't remember L'Hôpital's rule from your previous calculus calss, Stewart's book has a nice treatment of it. You should also feel free to come to office hours and ask about it!

exists, we say that *the improper integral of f on $[a,\infty)$ converges* and write

$$\int_a^\infty f(t)\,dt = \lim_{x\to\infty} I(x) = \lim_{x\to\infty} \int_a^x f(t)\,dt.$$

The number $\int_a^\infty f(t)\,dt$ is said to be the improper integral of f on $[a,\infty)$.

Let's work out an example. Consider the function $f : [1,\infty) \to \mathbb{R}$ defined by

$$f(t) = \frac{1}{t^2}$$

for $t \geq 1 = a$. Our goal is to understand whether or not the improper integral

$$\int_1^\infty f(t)\,dt = \int_1^\infty \frac{1}{t^2}\,dt$$

converges. Of course, the function f is continuous on the interval $[1,\infty)$ and so it is ripe for this type of investigation. To sort things out, for any number $x > 1$, we compute

$$I(x) = \int_1^x \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]_1^x = 1 - \frac{1}{x}.$$

In this case,

$$\lim_{x\to\infty} I(x) = \lim_{x\to\infty} \left(1 - \frac{1}{x}\right) = 1 - 0 = 1\,.$$

Therefore, we can conclude that the improper integral in question converges and

$$\int_1^\infty \frac{1}{t^2}\,dt = \lim_{x\to\infty} I(x) = 1.$$

Now it's your turn:

a. By the definition above, determine whether or not the following improper integrals converge and, if so, give their value.

(i)
$$\int_1^\infty \frac{1}{t^{3/2}}\,dt$$

(ii)
$$\int_1^\infty \frac{1}{t}\,dt$$

(iii)
$$\int_0^\infty e^{-t}\,dt$$

(iv)
$$\int_0^\infty g(t)\,dt \qquad \text{if} \qquad g(t) = \begin{cases} t^2 & 0 \leq t \leq 4 \\ 0 & 4 < t < \infty \end{cases}$$

b. Use properties of limits (from Calc 1) to prove/justify the following facts. Together, these two facts illustrate that improper integration is linear (in some sense).

Fact A. *Let h and k be continuous real-valued functions on the interval $[a, \infty)$. If the improper integrals*

$$\int_a^\infty h(t)\,dt \qquad \text{and} \qquad \int_a^\infty k(t)\,dt$$

converge, then the associated improper integral of $h+k$ on $[a, \infty)$ converges and

$$\int_a^\infty (h(t)+k(t))\,dt = \int_a^\infty h(t)\,dt + \int_a^\infty k(t)\,dt\,.$$

Fact B. *Let h be a continuous real-valued function on the interval $[a, \infty)$ and let c be any real number. If the improper integral*

$$\int_a^\infty h(t)\,dt$$

converges, then the associated improper integral of ch on $[a, \infty)$ converges and

$$\int_a^\infty ch(t)\,dt = c\int_a^\infty h(t)\,dt\,.$$

Problem 6 (Some interesting ways to find limits)**.** In this problem, we will study a couple of interesting ways to find limits.

a. Consider the sequence $\{a_n\}$ defined by

$$a_n = \sum_{k=1}^{n} \frac{k}{n^2}$$

for $n = 1, 2, \ldots$.

(i) Find the values of a_5 and a_{10} of this sequence.

(ii) Thinking back to Calculus 1 and, in particular, Riemann integration and Riemann sums, explain why $\{a_n\}$ must be a convergent sequence and[5]

$$\lim_{n\to\infty} a_n = \int_0^1 x\,dx = \frac{1}{2}.$$

b. Consider the sequence $\{b_n\}$ defined by

$$b_n = \frac{1}{n^2}\sin\left(n\frac{\pi}{12}\right)$$

for $n = 1, 2, \ldots$.

(i) Find the values of b_3 and b_6 of this sequence.

(ii) Does the sequence $\{b_n\}$ converge? If your answer is "no", explain your reasoning. If your answer is "yes", find the value of the limit. (Hint: You might want to play around with the inequality $-1 \le \sin(\theta) \le 1$ for all θ.)

Problem 7 (Application of sequences)**.** Analysis of sequences and their convergence (or not!) is a huge part of applied/computational mathematics. In your previous math courses, you have encountered problems that can solved using pen and paper, but in many real-world applications this is not the case. When solutions cannot easily be solved for, we use algorithms and numerical methods to generate a sequence of "guesses" called iterates, $\{x_k\}$, until a "good enough" guess is output. This problem will examine a method for approximating the roots[6] of a function, so that $f(x_k) \approx 0$.

[5] Hint: $k/n^2 = (k/n)\cdot(1/n)$.

[6] Recall that, for a given real-valued function f, a root of f is a point x for which $f(x) = 0$.

a. Consider the function $f(x) = x^2 + 3x - 1$. This function cannot easily be factored, so let's explore using a root finding method to figure out the roots, instead.

(i) Let $x_0 = 1$. Find the linear approximation, $L(x)$, to f at x_0.

(ii) Find the root of the linear approximation $L(x)$ from (i), i.e., the value x_1 such that $L(x_1) = 0$.

(iii) Find the linear approximation and the root, x_2, of the linear approximation at x_1.

(iv) Using a software of your choice or the quadratic formula, find the roots of f. Is x_2 a good approximation of a root?

b. Now, consider a differentiable function $f(x)$ and point x_0.

(i) Write the equation for the linear approximation of f at x_0.

(ii) Let x_1 be the root of the linear approximation of f at x_0. Find the equation for x_1.

(iii) By thinking *iteratively*, can you give a formula for the approximate root x_k in terms of the previous one, x_{k-1}?

c. This method, by which one approximates roots of a function by the sequence $\{x_k\}$, is called Newton's Method for root finding, and it relies on the linear approximation. This week, we introduced the idea of Taylor polynomials which give us degree n polynomial approximations for every n.

(i) In part (a), we looked at approximating roots of a quadratic function using a sequence of linear approximations (degree 1 Taylor polynomials). What if we wanted to use the degree 2 Taylor polynomial? Does it make sense to approximate a quadratic function with a degree 2 polynomial?

(ii) A linear approximation gives us a line which has *exactly* one real root (as long as it's not a horizontal line). Higher degree polynomials have more roots. Why does this complicate the idea of Newton's method?

(iii) Is a higher degree polynomial always better?