# Predicates and Quantifiers

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## 1 Predicates

We use the terms "statement" and "claim" interchangeably.

### 1.1 Definition

A **(mathematical) predicate in one variable** is a mathematical sentence involving a "free variable parameter" which ranges over a set of values, referred to as *the domain of the predicate*.

If the variable in the predicate is replaced by a value, the resulting sentence is a mathematical statement (and hence is either true or false).

One may think of a predicate as a *collection of statements*, one for each possible value of the variable. We say that the statements are "*indexed*" by the variable.

### 1.2 Example

"x+3=2" is a predicate when $x$ varies over $\mathbb{R}$. When $x = 3.1$ we get a false statement; when $x = -1$ we get a true one; etc.

### 1.3 Example

A bit trickier:

$$\text{"if } x^2 > \sin(x) \text{ then } \cos(x) > x^3\text{"}$$

is also an example of a predicate. When $x = 0$ it yields a true statement, and when $x = 1$ – a false one.

Note that the hypothesis and the conclusion in this "conditional" predicate are themselves predicates!

Don't forget that a *conditional statement* is false only when the hypothesis is a true statement and the conclusion is a false one.

### 1.4 Definition

**Predicates in several variables** are defined similarly. For example, when $x$ and $y$ range over $\mathbb{R}$, "$x^2 + y^2 = 1$" is a predicate in two variables.

One can also treat "$x^2 + y^2 = 1$" as a predicate in three variables $x, y, z$, with $x, y, z$ ranging over $\mathbb{R}$, even though $z$ is not visible (think of it as being present but with a zero coefficient, as in $x^2 + y^2 + 0z = 1$).

It is common to drop off the quotation marks around statements and predicates, and to rely on other forms of delineation, or on the context. We shall do so henceforth.

Usually one may infer the domains of the variables of a predicate implicitly from the context, similarly to the way this is done for the domains of functions. For example, in looking at $\frac{1}{x} + \sqrt{y} = z$ within $\mathbb{R}$ one will assume that $x$ is non-zero and $y$ is non-negative.

### 1.5 Notation

We write $P(x)$ or $S(t)$ to indicate a predicate in variable $x$, and a predicate in variable $t$ respectively. Obviously $T(t, x, \#)$ is used to indicate a predicate in variables $t, x, \#$.

If one thinks of a predicate as signifying a collection of statements, one can immediately see that *the names for the variables can be altered consistently throughout, without changing the predicate*. If $x$ and $w$ range over $\mathbb{Q}$, and $y$ and $u$ range over $\mathbb{R}$, then the predicates

$$x^2 - y^3 < xy$$

and

$$w^2 - u^3 < uw$$

are *identical* because they represent exactly the same collections of statements. Let us adopt this point of view.

### 1.6 Comment

Note that in Chapter 3, Biggs is playing rather loose with these concepts. To correct for this imprecision simply interpret his $p$'s, $q$'s and $r$'s as predicates, especially when he starts constructing truth tables.

Biggs says that two statements are "equivalent" when he means that they have the same truth value. We will say the latter, not the former. "Logically equivalent" shall be reserved for the realm of predicates.

Notice that in 3.6 Biggs does use predicates when he talks about universal and existential quantification.

### 1.7 Definition

"**Quantified statements**" are statements about predicates (i.e. about collections of statements). For example,

$$\forall x \in \mathbb{R} : x^2 > 0$$

The symbol "$\forall$" is read as "for all" or "for every".

is a claim that every statement of the form $\star^2 > 0$ (with $\star$ a real number) is true. On the other hand,

$$\exists x \in \mathbb{R} \text{ such that } (\text{if } x^2 > \sin(x) \text{ then } \cos(x) > x^3) \quad (1)$$

The symbol "$\exists$" is read as "there exists".

Is claim (1) true?

is the claim that for at least one real value $a$ of $x$ the statement

$$\text{if } a^2 > \sin(a) \text{ then } \cos(a) > a^3$$

is true.

Similarly

$$\forall x \in \mathbb{R} : (\exists y \in \mathbb{R} \text{ such that } x^3 + \sin y > 0) \quad (2)$$

Is claim (2) true?

is a claim that no matter what real value $\alpha$ is assigned to $x$ (first!), there is a real value $\beta$ for $y$ such that $\alpha^3 + \sin \beta > 0$ is true.

In general, if $P(x)$ is a predicate and $x$ ranges over domain $D$, then

$$\forall x \in D : P(x)$$

and

$$\exists x \in D \text{ such that } P(x)$$

are statements (NOT predicates!). The first one is true exactly when every statement in the collection signified by $P(x)$ is true; the second – when at least one statement in the collection is true.

You may have noticed in your previous encounters with mathematical texts that in mathematics it is quite common to "drop off" a universal quantifier (i.e. to keep it implicit) when the context is clear.

For example, a proclamation

$$n^2 + n \text{ is an even integer}$$

A "proclamation" is a claim that a statement is true.

would be one such instance. Since

$$n^2 + n \text{ is an even integer}$$

is a predicate, it has no inherent truth value.

The actual proclamation being made here is in fact:

$$n^2 + n \text{ is } \textit{always} \text{ an even integer;}$$

or, more prcisely:

$$\text{for any integer } n : n^2 + n \text{ is an even integer;}$$

or, in other words:

$$\forall n \in \mathbb{Z} : 2 \text{ divides } n^2 + n.$$

Notice that the implicit domain of the variable $n$ here is taken to be $\mathbb{Z}$. This is the most general domain based on our interpretation of the naming convention for the variables, and the terminology of the predicate.

To turn a predicate into a statement through a quantification of the variables, one needs to quantify *all* of the variables.

For example,

Is (3) a true statement?

$$\forall x \in \mathbb{R} : (\exists y \in \mathbb{R} \ \text{ such that } \ x^3 + \sin y > 0) \tag{3}$$

is a statement, while

$$\exists y \in \mathbb{R} \ \text{ such that } \ x^3 + \sin y > 0 \tag{4}$$

is a predicate in variable $x$, and NOT a statement! Indeed, for each fixed value of $x$, (4) makes a claim about a collection of statements parametrized by $y$. This claim is either true or false, depending on which value of $x$ is chosen at the outset. So, (4) is a collection of claims parametrized by $x$.

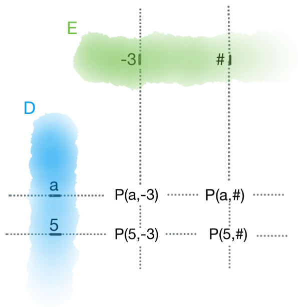
Quantified statements are read from left to right. Still, one often hears folks refer to the "outside" quantifiers and the "inside" quantifiers. This may remind you of the way Chain Rule procedure was described in your Calculus course.

In general, if $P(x, y)$ is a predicate and $x$ ranges over domain $D$, and $y$ ranges over $E$, then there are eight natural (doubly-)quantified claims that can be made:

★★★

1. $\forall x \in D : \big(\forall y \in E : P(x, y)\big)$
2. $\forall x \in D : \big(\exists y \in E \ \text{ such that } \ P(x, y)\big)$
3. $\exists x \in D \ \text{ such that } \ \big(\exists y \in E \ \text{ such that } \ P(x, y)\big)$
4. $\exists x \in D \ \text{ such that } \ \big(\forall y \in E : P(x, y)\big)$
5. $\forall y \in E : \big(\forall x \in D : P(x, y)\big)$
6. $\forall y \in E : \big(\exists x \in D \ \text{ such that } \ P(x, y)\big)$
7. $\exists y \in E \ \text{ such that } \ \big(\exists x \in D \ \text{ such that } \ P(x, y)\big)$
8. $\exists y \in E \ \text{ such that } \ \big(\forall x \in D : P(x, y)\big)$

To visualize the kinds of claims these statements represent, let us imagine that all of the statements signified by the predicate $P(x, y)$ are arranged in a table of sorts, with values of $x$ indicating the rows of the table (ex. $x_o$-th row) and the values of $y$ – the columns, as per diagram below.




For example, if our predicate P(x,y) is

$$x^3 + \sin y > 0,$$

with $x$ ranging over the rational numbers, and $y$ ranging over the reals, then the statement in the $-\frac{2}{11}$th row and $\pi$th column of the "$P$-table" is

$$-\frac{8}{11^3} + \sin \pi > 0.$$

Now, the claim

$$\forall x \in D : (\forall y \in E : P(x, y))$$

is true exactly when in every row of the $P$-table every statement is true. In other words, exactly when all statements in the table are true.

The claim

$$\forall x \in D : (\exists y \in E \text{ such that } P(x, y))$$

holds true exactly when in every row of the $P$-table at least one of the statements is true. Therefore, the claim is false exactly when there is a row of the $P$-table which holds only false statements.

### 1.8 Exercise

1. Work through the remaining five statements on the list ★★★ above, and for each write down what it means for it to be true (and then – false) in terms of the truth values of the statements (and their locations) in the corresponding $P$-table.
2. Argue that the statements 1) and 5) on the list ★★★ above always have the same truth value.
3. Argue that the statements 3) and 7) on the list ★★★ above always have the same truth value.

# 2 Negating quantified statements

A few minutes of thought should convince you of the validity of the following method of negating quantified statements when only one variable is involved:

Symbol "¬" is read as "it is not the case that"

Convince yourself that when $P(x)$ is a predicate, so is $\neg P(x)$.

$$\neg(\forall x \in \mathbb{R} : P(x)) \tag{5}$$

has the same truth value as

$$\exists x \in \mathbb{R} \text{ such that } \neg P(x). \tag{6}$$

Indeed, (5) is true exactly when

$$\forall x \in \mathbb{R} : P(x)$$

is false, i.e. exactly when there is at least one real value $a$ of $x$ such that $P(a)$ is false; i.e. exactly when (6) is true.

Similarly

$$\neg(\exists x \in \mathbb{R} \text{ such that } P(x))$$

has the same truth value as

$$\forall x \in \mathbb{R} : \neg P(x).$$

The same principle applies when more than one variable and quantifier are involved. For example,

One wacky way to remember how this works is to imagine that a negatory wizard Voldemort, pointing the Elder wand $\neg$, is fighting his way from the left into the claim Hogwarts towards Harry Potter predicate $P(x,y)$. Each time you-know-who encounters a quantifier manned by a guardian variable, he issues a spell and turns the quantifier into its counterpart ($\forall$ to $\exists$, and vice versa), thus discombobulating the variable long enough to pass by. The wizard stops when he reaches Harry Potter predicate $P(x,y)$ and points the wand to Harry's chest.

$$\neg\,(\forall x \in D : (\exists y \in E \text{ such that } P(x,y)))$$

has the same truth value as

$$\exists x \in D \text{ such that } (\forall y \in E : \neg P(x,y))$$

# 3 Exercises

### 3.1 Exercise

Consider the following two statements.

1. "$\forall a \in \mathbb{R} : \Big(a > 0 \Longrightarrow a^{113} - a^{100} + 0.123 > 0\Big)$".
2. "$\forall a \in (0, \infty) : a^{113} - a^{100} + 0.123 > 0$".

Argue that these statements have the same truth value. To this end you need to demonstrate that when of the statements is true, then so is the other. (Why is this sufficient?)

### 3.2 Exercise

In this exercise you are being asked to extend the result of Exercise 3.1.

Suppose that $P(\gamma)$ is a predicate with $\gamma$ ranging over $\mathbb{Z}$. Here (and elsewhere) $\mathbb{N}$ stands for the set of positive integers, and $\mathbb{Z}$ stands for the set of all integers.

Argue that the following statements have the same truth value.

1. $\forall \gamma \in \mathbb{Z} : \Big(\ \gamma \in \mathbb{N} \Longrightarrow P(\gamma)\ \Big)$.
2. $\forall \gamma \in \mathbb{N} : P(\gamma)$.

### 3.3 Exercise

In this exercise you are being asked to extend the result of Exercises 3.1 and 3.2.

Suppose that $P(\gamma)$ is a predicate with $\gamma$ ranging over a set $\mathcal{T}$, that contains a set $\mathcal{S}$.

Argue that the following statements have the same truth value.

1. $\forall \gamma \in \mathcal{T} : \Big(\ \gamma \in \mathcal{S} \Longrightarrow P(\gamma)\ \Big)$.
2. $\forall \gamma \in \mathcal{S} : P(\gamma)$.

### 3.4 Exercise

Take $P(n, w)$ to be the sentence " $nw \geq 10^6$ " where $n$ ranges over $\mathbb{N}$ and $w$ ranges over the open interval $(0, 1)$ of real numbers, and decide which of the eight natural doubly-quantified claims on the list ★★★ above are true.

### 3.5 Exercise

Find a predicate $P(x,y)$ with $x$ and $y$ ranging over $\mathbb{N}$, such that BOTH of the following are satisfied:

1. "$\forall x \in \mathbb{N} : (\exists y \in \mathbb{N} \text{ such that } P(x,y))$" is a true statement;
2. "$\exists y \in \mathbb{N} \text{ such that } (\forall x \in \mathbb{N} : P(x,y))$" is a false statement.

### 3.6 Exercise

By giving concrete examples of predicates, demonstrate that apart from the pairs indicated in Exercise 1.8, no other pairs of statements on the list ★★★ always have identical truth values.

### 3.7 Exercise

Consider the proverb:

> "One can fool some of the people all of the time and all of the people some of the time, but one cannot fool all of the people all of the time."

1. Let $M(\rho,t)$ stand for

   "One can fool person $\rho$ at the time instance $t$."

   Let us agree that $t$ ranges over the set $\mathcal{T}$ of all moments of time, and $\rho$ ranges over the set $\mathcal{P}$ of all people on earth. Fill in "..." appropriately to obtain a representation of the proverb above in the following form:

$$\Big(\exists \rho \in \mathcal{P} \text{ such that } ( \ \dots \ M(\rho,t))\Big) \text{ and }$$
$$\Big(\exists t \in \mathcal{T} \text{ such that } ( \ \dots \ M(\rho,t))\Big) \text{ and } \Big(\neg(\forall \ \dots \ M(\rho,t))\Big)$$

2. Use the result of the first part of the problem and the methods of negation of quantified statements to write down the NEGATION of the proverb in English. Obviously I am NOT after something which uses expressions such as "it is not the case". Your answer should be along the lines of the original statement, and should sound good and perhaps be even wise.

### 3.8 Exercise

1. Suppose that $P(x)$ is a predicate with $x$ varying over a given set $\Omega$, and $Q$ is a statement. Argue that the statements

$$\Big(\exists x \in \Omega \ \text{ such that } \ P(x)\Big) \vee Q$$

and

$$\exists x \in \Omega \ \text{ such that } \ \Big(P(x) \vee Q\Big)$$

have the same truth value.

2. The following paradox is based on an idea by Timothy Gowers.
It should be obvious to you that the following claim is true for any subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{R}$:

$$\Big(\forall x \in \mathcal{S} : x \in \mathcal{T}\Big) \implies \mathcal{S} \subset \mathcal{T}. \tag{7}$$

To simplify the notation, let us write $\Box(x)$ for "$x \in \mathcal{T}$", and $\triangle$ for "$\mathcal{S} \subset \mathcal{T}$". We will carry along the universal quantification over all subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{R}$ implicitly.
The true statement (7) can then be expressed as

$$\Big(\forall x \in \mathcal{S} : \Box(x)\Big) \implies \triangle,$$

and (by the definition of the connective "$\implies$") has the same truth value as the statement

$$\neg\Big(\forall x \in \mathcal{S} : \Box(x)\Big) \vee \triangle,$$

which, in turn, has the same truth value as

$$\Big(\exists x \in \mathcal{S} \ \text{ such that } \ \neg\Box(x)\Big) \vee \triangle.$$

Applying the result of part 1 to this last true statement, we can conclude that the statement

$$\exists x \in \mathcal{S} \ \text{ such that } \ \Big(\neg\Box(x) \vee \triangle\Big)$$

is true, and therefore the statement

$$\exists x \in \mathcal{S} \ \text{ such that } \ \Big(\Box(x) \implies \triangle\Big)$$

is true. Returning to the original notation, and making the quantification explicit yields a true statement

Timothy Gowers is a British mathematician who received the Fields Medal, a highest prize in mathematics, in 1998, but not for this paradox. :-)

To say that one set ($\mathcal{S}$) of real numbers is **a subset** of another ($\mathcal{T}$), is to say that every element of $\mathcal{S}$ is an element of $\mathcal{T}$. We write $\mathcal{S} \subset \mathcal{T}$ for the statement "$\mathcal{S}$ is a subset of $\mathcal{T}$".

*For any subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{R}$: $\exists x \in \mathcal{S}$ such that*
$$\Big(x \in \mathcal{T} \Longrightarrow \mathcal{S} \subset \mathcal{T}\Big).$$

Expressing this in an even less compressed form we have:

*For any subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{R}$: there exists an element $x$ of $\mathcal{S}$ such that* $\Big(x \in \mathcal{T} \Longrightarrow \mathcal{S} \subset \mathcal{T}\Big)$.

Yet it is clear that this last statement cannot possibly be true, since expressed in a more plain language it claims that for any pair of subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{R}$, there is always a "special" element of $\mathcal{S}$ such that its membership in $\mathcal{T}$ automatically entails the membership of all other elements of $\mathcal{S}$ in $\mathcal{T}$ (i.e. $\mathcal{S} \subset \mathcal{T}$).
Find the flaw in the argument and resolve the paradox.