1.11 Theorem *Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then*

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$.

In particular, $\inf B$ *exists in S.*

Proof Since B is bounded below, L is not empty. Since L consists of exactly those $y \in S$ which satisfy the inequality $y \le x$ for every $x \in B$, we see that *every $x \in B$ is an upper bound of L*. Thus L is bounded above. Our hypothesis about S implies therefore that L has a supremum in S; call it α.

If $\gamma < \alpha$ then (see Definition 1.8) γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $\alpha \le x$ for every $x \in B$. Thus $\alpha \in L$.

If $\alpha < \beta$ then $\beta \notin L$, since α is an upper bound of L.

We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$. In other words, α is a lower bound of B, but β is not if $\beta > \alpha$. This means that $\alpha = \inf B$.

FIELDS

1.12 Definition A *field* is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

(A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.
(A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
(A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
(A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
(A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0.$$

(M) Axioms for multiplication

(M1) If $x \in F$ and $y \in F$, then their product xy is in F.
(M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
(M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
(M4) F contains an element $1 \ne 0$ such that $1x = x$ for every $x \in F$.
(M5) If $x \in F$ and $x \ne 0$ then there exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1.$$

(D) The distributive law

$$x(y+z) = xy + xz$$

holds for all $x, y, z \in F$.

1.13 Remarks

(*a*) One usually writes (in any field)

$$x - y, \frac{x}{y}, x + y + z, xyz, x^2, x^3, 2x, 3x, \ldots$$

in place of

$$x + (-y), x \cdot \left(\frac{1}{y}\right), (x+y)+z, (xy)z, xx, xxx, x+x, x+x+x, \ldots.$$

(*b*) The field axioms clearly hold in Q, the set of all rational numbers, if addition and multiplication have their customary meaning. Thus Q is a field.

(*c*) Although it is not our purpose to study fields (or any other algebraic structures) in detail, it is worthwhile to prove that some familiar properties of Q are consequences of the field axioms; once we do this, we will not need to do it again for the real numbers and for the complex numbers.

1.14 Proposition *The axioms for addition imply the following statements.*

(*a*) *If* $x + y = x + z$ *then* $y = z$.
(*b*) *If* $x + y = x$ *then* $y = 0$.
(*c*) *If* $x + y = 0$ *then* $y = -x$.
(*d*) $-(-x) = x$.

Statement (*a*) is a cancellation law. Note that (*b*) asserts the uniqueness of the element whose existence is assumed in (A4), and that (*c*) does the same for (A5).

Proof If $x + y = x + z$, the axioms (A) give

$$\begin{aligned} y = 0 + y = (-x + x) + y &= -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$

This proves (*a*). Take $z = 0$ in (*a*) to obtain (*b*). Take $z = -x$ in (*a*) to obtain (*c*).

Since $-x + x = 0$, (*c*) (with $-x$ in place of x) gives (*d*).

1.15 Proposition *The axioms for multiplication imply the following statements.*

(*a*) *If* $x \neq 0$ *and* $xy = xz$ *then* $y = z$.
(*b*) *If* $x \neq 0$ *and* $xy = x$ *then* $y = 1$.
(*c*) *If* $x \neq 0$ *and* $xy = 1$ *then* $y = 1/x$.
(*d*) *If* $x \neq 0$ *then* $1/(1/x) = x$.

The proof is so similar to that of Proposition 1.14 that we omit it.

1.16 Proposition *The field axioms imply the following statements, for any* x, y, $z \in F$.

(*a*) $0x = 0$.
(*b*) *If* $x \neq 0$ *and* $y \neq 0$ *then* $xy \neq 0$.
(*c*) $(-x)y = -(xy) = x(-y)$.
(*d*) $(-x)(-y) = xy$.

Proof $0x + 0x = (0 + 0)x = 0x$. Hence 1.14(*b*) implies that $0x = 0$, and (*a*) holds.

Next, assume $x \neq 0$, $y \neq 0$, but $xy = 0$. Then (*a*) gives

$$1 = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)xy = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)0 = 0,$$

a contradiction. Thus (*b*) holds.

The first equality in (*c*) comes from

$$(-x)y + xy = (-x + x)y = 0y = 0,$$

combined with 1.14(*c*); the other half of (*c*) is proved in the same way. Finally,

$$(-x)(-y) = -[x(-y)] = -[-(xy)] = xy$$

by (*c*) and 1.14(*d*).

1.17 Definition An *ordered field* is a *field* F which is also an *ordered set*, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
(ii) $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, and $y > 0$.

If $x > 0$, we call x *positive*; if $x < 0$, x is *negative*.

For example, Q is an ordered field.

All the familiar rules for working with inequalities apply in every ordered field: Multiplication by positive [negative] quantities preserves [reverses] inequalities, no square is negative, etc. The following proposition lists some of these.

1.18 Proposition *The following statements are true in every ordered field.*

(*a*) *If* $x > 0$ *then* $-x < 0$, *and vice versa.*
(*b*) *If* $x > 0$ *and* $y < z$ *then* $xy < xz$.
(*c*) *If* $x < 0$ *and* $y < z$ *then* $xy > xz$.
(*d*) *If* $x \neq 0$ *then* $x^2 > 0$. *In particular,* $1 > 0$.
(*e*) *If* $0 < x < y$ *then* $0 < 1/y < 1/x$.

Proof
(*a*) If $x > 0$ then $0 = -x + x > -x + 0$, so that $-x < 0$. If $x < 0$ then $0 = -x + x < -x + 0$, so that $-x > 0$. This proves (*a*).
(*b*) Since $z > y$, we have $z - y > y - y = 0$, hence $x(z - y) > 0$, and therefore

$$xz = x(z - y) + xy > 0 + xy = xy.$$

(*c*) By (*a*), (*b*), and Proposition 1.16(*c*),

$$-[x(z - y)] = (-x)(z - y) > 0,$$

so that $x(z - y) < 0$, hence $xz < xy$.
(*d*) If $x > 0$, part (ii) of Definition 1.17 gives $x^2 > 0$. If $x < 0$, then $-x > 0$, hence $(-x)^2 > 0$. But $x^2 = (-x)^2$, by Proposition 1.16(*d*). Since $1 = 1^2$, $1 > 0$.
(*e*) If $y > 0$ and $v \leq 0$, then $yv \leq 0$. But $y \cdot (1/y) = 1 > 0$. Hence $1/y > 0$. Likewise, $1/x > 0$. If we multiply both sides of the inequality $x < y$ by the positive quantity $(1/x)(1/y)$, we obtain $1/y < 1/x$.

THE REAL FIELD

We now state the *existence theorem* which is the core of this chapter.

1.19 Theorem *There exists an ordered field R which has the least-upper-bound property.*

Moreover, R contains Q as a subfield.

The second statement means that $Q \subset R$ and that the operations of addition and multiplication in R, when applied to members of Q, coincide with the usual operations on rational numbers; also, the positive rational numbers are positive elements of R.

The members of R are called *real numbers.*

The proof of Theorem 1.19 is rather long and a bit tedious and is therefore presented in an Appendix to Chap. 1. The proof actually constructs R from Q.