Manipulating Nameless Numbers?

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I will take it for granted that you are familiar with the Arabic numerals denoting whole numbers, including the relative newcomers: the negative numbers and zero, as well as the basic algebraic operations the humanity has designed for manipulating these numbers.

For example, "$4+2$" by definition refers to the same object as "6", as does "$2 \cdot 3$," etc.

Here is a small philosophical conundrum for you to consider: since there are infinitely many integers, and humanity has only existed for a finite number of seconds, most integer names have never been written, pronounced, or even thought of by anybody. Yet collectively we firmly believe that we have a system by which we have named all possible integers!

While rational numbers go by many different "names" (ex. $\frac{2}{3}, \frac{6}{9}, \frac{14}{21}$), and we have a system by which all are named, most irrational numbers have no names at all. You have been led to believe that the (standard) Real Number System contains all fractions and also, among others, a positive number which squares to 2. Such a number was then given a name: "$\sqrt{2}$ ", and it turns out that this number is not a fraction.

Many other real numbers can be described through some algebraic properties as well, but most irrational numbers are out of luck: there is no standard naming system that somehow corrals them all. What is worse, there is no possibility of inventing such a system. As you will see in this class there are far too many irrational numbers for such a naming system to exist, many more irrational numbers (by far!) than there are integers, or even fractions.

At this point, some may enter into an discussion as to what is meant by a "naming system". Not to belabor the point, let us say that "a naming system" here is a system which involves some <u>finite</u> "alphabet" of symbols, out of which finite "words" are formed to be used as "the names" for the objects at hand. Please convince yourself that in this sense we do indeed have a naming system for all integers and for all fractions!

I am hoping that you are now a bit more puzzled about "the Real Number System". Supposedly it contains as many "numbers" as there are points on a straight line, and supposedly there are operations (commonly referred to as "the addition" and "the multiplication" by analogy with the operations on the integers) to combine such numbers to produce other such numbers.

Yet most “real numbers” have no names, and so there is no way to specify these algebraic operations in any manner similar to the way this was done for the fractions.

How can we then be so sure that “the Real Number System” with all of its wonderful properties is not just a pipe dream? What properties, you ask? Well, all those you have been using liberally since the very first time you started studying Calculus. For example, you wouldn’t probably think twice before writing $\sqrt[37]{2}$, or even working with a number symbolically described by a letter “e”, and writing

$$e^{\sqrt[37]{2}}.$$

Every time you remained assured that there was a real number your notation referred to.

On the other hand, you refrained from writing $\sqrt{-1}$, because through some argument you were convinced that no real number squares to -1. Yet I wonder if any argument was presented to affirm your belief that there *is* a real number such that by multiplying it with itself 37 times you would get 2 as an answer.

Believe it or not, “the Real Number System” <u>was</u> a kind of a dream (or more precisely, a near-religious belief) for the humanity up until a mathematician Richard Dedekind (1831-1916) was able to find a way to actually construct such a system. Does it surprise you that Newton, an inventor of Calculus, was actually working with a potentially non-existent numbering system? Well, it should!

What is more, Newton actually worked with a very different numbering system that included “infinitely small positive numbers” (a.k.a. “infinitesimals” or “fluxions”), and it was not until the middle of the 20-th century that a mathematician Abraham Robinson was able to set such a system on a rigorous foundation.

In our course we will not use Robinson’s (and Newton’s) real number system, but instead will follow the lead of Augustin-Louis Cauchy, who in 1821 had laid a firm foundation for Calculus, re-casting the ideas of Lebnitz and Newton (via the concept of “a limit”) into a form with which you are familiar. This was the form introduced in you previous Calculus course.

Of course, since Cauchy predates Dedekind, his theory was also founded on a belief that there is indeed an underlying real number system with the properties required for his rigorous casting of the concepts of Calculus. Fortunately, the common intuition that supported such a belief proved to be right, which, alas, is not always the case (for example, the “flat earth” theory and the theory of “aether” were long standing theories supported by common intuition that proved to be quite wrong).

So what kind of properties of the real number system were essential for the rigorous development of calculus, and why was the rational number system insufficient for that purpose? This deep question is the starting point for our journey. Bon voyage!