Math 135: Homework 8 – not to be turned in

Below are four exercises that I would like you to do before Wednesday's midterm exam. As there won't be time to grade them and give you solution feedback, I will make solutions available to you on Monday. To reward you for doing these exercises before you receive solutions, please come to class on Monday with your solutions/attempts, show them to me, and I will give you some credit. In the meantime, if you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. In class, we discussed a version of the ratio test that is different from the one you see in Abbott's exercises and what the students in MA160 see. In particular, we proved the following theorem.

Theorem A. *Let $A = (a_n)$ be a sequence and consider the series $\sum a_n$. We have:*

1. *If*
$$\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1,$$
then the series $\sum a_n$ converges absolutely.
2. *If, for some $N \in \mathbb{N}$,*
$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1$$
for all $n \geq n_0$, the the series diverges.

The students in MA160 see the following version of the ratio test:

Theorem B. *Let $A = (a_n)$ be a sequence of real numbers and consider the series $\sum a_n$. Assume that the limit*
$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$
exists.

1. *If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
2. *If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

In this exercise, you will explore these theorems and prove, in particular, that our theorem (Theorem A) is stronger than Theorem B.

1. In your own words, explain one way in which the hypotheses of Theorem B is more restrictive than that required in Theorem A.
2. Show that the fist statement of Theorem B follows from the first statement of Theorem A.
3. Find a situation where the limit defining L (in Theorem B) does not exist yet Statement 1 of Theorem A still applies and guarantees absolute convergence.
4. Show that the second statement of Theorem B follows from the second statement of Theorem B.
5. Find a situation where the second conclusion of Theorem B does not hold, however the second condition of Theorem A does hold (and so Theorem A) applies.
6. Finally, if you look back at your lecture notes, I didn't prove Part 2 of Theorem A. Prove it. Hint: Use the divergence test.

Exercise 2. In class, we studied Hadamard's theorem. Here, I will give you the full statement (which is written in terms of roots and not ratios).

Theorem C (Hadamard)**.** *Given a sequence of real numbers $A = (a_n)$, consider the power series*

$$\sum_{n=0}^{\infty} a_n x^n$$

and define

$$1/r = \limsup |a_n|^{1/n}$$

which always exists (though it could be infinity and in this case we take $r = 0$). We have three cases:

1. *The case $r = 0$: In this case, the series converges only at $x = 0$.*
2. *The case $r = \infty$: In this case, the series converges absolutely on the entire real line $\mathbb{R} = (-\infty, \infty)$.*
3. *The case that r is positive and finite. In this case, the series converges on one of the following intervals:*

   $$(-r, r), \qquad [-r, r] \qquad (-r, r], \qquad \textit{or} \qquad [-r, r)$$

   In all cases, the convergence is absolute on the "interior" $(-r, r)$ but anything can happen at the endpoints.

Note: What happens at the endpoints must be checked by simply plugging in $x = \pm r$ and determining convergence. Please do the following:

1. Show that the series $\sum_{n=0}^{\infty} x^n$ has radius of convergence $r = 1$ but converges only on the interval $(-1, 1)$ (and not at the endpoints). Do you know a function $f(x)$ that is, in this case, a rational function, which can be represented by the sum

   $$f(x) = \sum_{n=0}^{\infty} x^n$$

   for $x \in (-1, 1)$. What is the function?
2. Show that the series $\sum_{n=0}^{\infty} x^n/n^n$ has radius of convergence $r = \infty$.
3. Show that the series $\sum_{n=0}^{\infty} x^n/n!$ has radius of convergence $r = \infty$. This will require you to show that

   $$\lim(n!)^{1/n} = \infty.$$

   In this way, you should observe that the so-called root test is much harder to apply than the ratio test.
4. Use the computation (and not the result) in the previous item, to show that the series

   $$\sum_{n=0}^{\infty} n! x^n$$

   has radius of convergence $r = 0$.
5. Find the intervals of convergence of the following series. If you know a function that represents the sum of the series, what is it?

   (a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$
   (b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$
   (c) $\sum_{n=0}^{\infty} n x^n$
6. Finally, though I haven't proven the above version of the theorem, it is useful to see why understand why exactly, the series cannot converge for $|x| > r$. Prove that, if $|x| > r$, then the series

   $$\sum_{n=0}^{\infty} a_n x^n$$

   must diverge. Hint: Use what you know of the limit superior and apply the divergence test.

Exercise 3. In Abbott's first edition, please do: Exercises 3.2.2., 3.2.3, 3.2.8, and 3.2.12. These are:

3.2.2 Let
$$B = \left\{ \frac{(-1)^n n}{n+1} : n = 1, 2, \ldots \right\}.$$

(a) Find the limit points of B.

(b) Is B a closed set?

(c) Is B an open set?

(d) Does B contain any isolated points?

(e) Find $\overline{B}$.

3.2.3 Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbb{Q}$.

(b) $\mathbb{N}$.

(c) $\{x \in \mathbb{R} : x > 0\}$.

(d) $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

(e) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$.

3.2.8 Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(a) Show that L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof of Theorem 3.2.12.

3.2.12 Decide whether the following statemetns are true of false. Provide counterexamples for those that are false and supply proofs for those that are true.

(a) For any set $A \subseteq \mathbb{R}$, $\overline{A}^c$ is open.

(b) If a set A has an isolated point, it cannot be an open set.

(c) A set A is closed if and only if $\overline{A} = A$.

(d) If A is a bounded set, then $s = \sup(A)$ is a limit point of A.

(e) Every finite set is closed.

(f) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.