Math 135: Spooky Homework 7

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, October 31st at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. In the second edition of Abbott's book, do Exercises 2.6.2 and 2.6.3.

Exercise 2. In this exercise, we study sequences that are not quite Cauchy and others that are super-duper Cauchy. Please do the following:

1. Consider the sequence $A = (a_n)$ where $a_n = \sqrt{n}$. Prove (directly using the ϵ-N definition) that
$$\lim_{n\to\infty} |a_{n+1} - a_n| = 0.$$
2. In view of what you did in the previous part, this sequence almost looks Cauchy because subsequent terms get arbitrarily close to each other, i.e., for very large n, $|a_{n+1} - a_n|$ is very small. By using the ϵ-N definition of Cauchy, show that the sequence $A = (a_n)$ is not Cauchy.
3. By appealing to either Lemma 2.6.3 or Theorem 2.6.4, give another proof that $A = (a_n)$ is not Cauchy.
4. Consider the following definition:

   Definition A. A sequence $B = (b_n)$ is super-duper Cauchy if,
$$|b_{n+1} - b_n| < \frac{1}{2^{n+1}}$$
   for all $n \in \mathbb{N}$.

   Note that, if B is super-duper Cauchy it does easily satisfy the property that we showed A to satisfy earlier, i.e.,
$$\lim_{n\to\infty} |b_{n+1} - b_n| = 0.$$
   Give an example of non-constant sequences that are:
   (a) Super-duper Cauchy
   (b) Not Cauchy and not super-duper Cauchy
   (c) Cauchy, but not super-duper Cauchy.
5. Prove that, if B is super-duper Cauchy, then it is Cauchy.
6. Could you come up with a more general condition of the form
$$|b_{n+1} - b_n| < c_n$$
   where (c_n) is some sequence (and not necessarily $c_n = 1/2^{n+1}$) that would guarantee that $\{b_n\}$ was Cauchy?

Exercise 3. Please do Exercises 2.7.2, 2.7.3, and 2.7.4 in the second edition of Abbott's book.

Exercise 4. Please do Exercises 2.7.7, 2.7.8, and 2.7.9 in the second edition of Abbott's book.