Math 135: Homework 6

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, October 24th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1 (Rational Approximation to Square Roots)**.** We have proved that square roots exist and are unique. That is, for each $a \geq 0$, there is a unique number $r \geq 0$ for which $r^2 = a$. This number is called the square root of a and denoted by $r = \sqrt{a}$. Okay, so we know square roots always exist – but how do we compute them? Your calculator seems to do something reasonable, but what exactly does it do? In this exercise, we will explore one way to approximate[1] square roots which is "fast" in the sense that the approximations converge quickly to $\sqrt{a}$.

In what follows, let $a > 1$ and define a sequence of real numbers (x_n), iteratively, by setting $x_0 = a$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \tag{1}$$

for $n \geq 1$. Please do the following

1. Prove that, for all $n = 0, 1, 2, \dots,$
$$x_n^2 > a.$$
Hint: Check that it is true for $n = 0$. Beyond 0, observe that
$$x_n^2 + 2a + \frac{a^2}{x_n^2} = \left(x_n - \frac{a}{x_n}\right)^2 + 4a.$$
2. Use the preceding in equality, which holds for all n to prove that (x_n) is a decreasing sequence.
3. Make use of the monotone convergence theorem (and the order limit theorem) to prove that (x_n) converges to a number $x \geq 1$.
4. Prove that $x = \sqrt{a}$.
5. If $a < 1$, does this construction still give $\lim x_n = x = \sqrt{a}$? Prove your assertions. *Hint: Go back through all previous steps, figure out how things should change, and modify them (and your arguments) appropriately.*

Note: On our course website, I have included my Matlab script which runs the above numerical "scheme" for you to play around with. If you want help getting it running, come and see me.

Exercise 2. Please do Exercise 2.4.6 (on the Arithmetic-Geometric Mean) out of the second edition of Abbott's book.

Exercise 3. Please do Exercise 2.4.7 (on the Limit Superior) out of Abbott's book (2nd edition).

Exercise 4. From Abbott's book (2nd edition), please do:

1. Exercise 2.4.8.
2. Exercise 2.4.9

Exercise 5. From Abbott's book (2nd edition), please do:

1. Exercise 2.5.1

[1]Note: For a positive number $a \in \mathbb{Q}$ which is not a perfect square, $\sqrt{a}$ is not rational and so the BEST your calculator can ever do is provide a rational approximation to it. What we do here, as it turns out, is not how modern computes approximate square roots but it is still a very effective method.

2. Exercise 2.5.2
3. Exercise 2.5.5

Exercise 6. From Abbott's book (2nd edition), please do:

1. Exercise 2.5.6
2. Exercise 2.5.7

Exercise 7. Please do Exercise 2.5.8 from Abbott's book (2nd edition). Also, discuss how this approach connects to the idea of the limit superior and limit inferior.

Exercise 8. Please do Exercise 2.6.1 from Abbott's book (2nd edition).