# Math 135: Homework 5

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, October 10th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

**Exercise 1** (The Absolute Value)**.** For $a \in \mathbb{R}$, define

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

This is called the absolute value of $a$.

1. Prove: For all $a, b \in \mathbb{R}$, $|ab| = |a||b|$. Hint: Check a bunch of cases and just apply the definition. So, e.g, if $a < 0$ and $b > 0$, then ....
2. Prove: For all $a \in \mathbb{R}$, $|-a| = |a|$.
3. Prove: For all $a \in \mathbb{R}$, $|a| = \sqrt{a^2}$. Hint: You must use what we proved in class about uniqueness of square roots.
4. Prove: For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$. Hint: Again, check cases.
5. Prove: For all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$
6. Prove: Let $a, b \in \mathbb{R}$. If, for every $\epsilon > 0$, $|a - b| < \epsilon$, then $a = b$.
7. Finally, observe that if $a, b, c \in \mathbb{R}$, we have

$$|a - b| = |a - c + c - b| = |(a - c) - (b - c)| \leq |a - c| + |b - c|.$$

   where we have made use of the fourth item above. Explain why this inequality is called the *triangle inequality*. Hint: This of what this would mean if $a, b, c$ were points in a plane (and not just on a line). There isn't anything to prove here, just an explanation.

**Exercise 2.** Please do:

1. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit:
   (a) $\lim \frac{1}{6n^2+1} = 0$.
   (b) $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$.
   (c) $\lim \frac{2}{n+3} = 0$.
2. Argue that the sequence

$$1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, -- \text{five 0's} --, 1, \ldots$$

   does not converge to zero. For what values of $\epsilon > 0$ does there exists a response $N$? For which values of $\epsilon > 0$ is there no response?
3. Please do Exercise 2.2.4 in the second edition of Abbott.

**Exercise 3.** Recall our definition of convergence:

**Definition A.** Let $(a_n)$ be a sequence of real numbers and $a$ another real number. We say that the sequence $a_n$ converges to $a$ if: For all $\epsilon > 0$, there is a natural number $N$ for which

$$|a_n - a| < \epsilon \qquad \text{whenever} \qquad n \geq N.$$

In this case we will call $a$ the limit of the sequence and write $\lim_n a_n = a$.

Okay, now the preceding definition pertains to a sequence converging to a particular value of $a$. It is useful to make the following definition so that we can just talk about the sequence itself.

**Definition B.** A sequence $(a_n)$ is said to converge if there exists $a \in \mathbb{R}$ such that $(a_n)$ converges to $a$. In this case, we call $a$ the limit of the sequence $(a_n)$. If $(a_n)$ does not converge, we say it diverges.

With these definitions in mind, please do the following:

1. In class, we proved that the sequence $(a_n)$ defined by

$$a_n = \frac{(-1)^n}{10^{10}}$$

   for $n \in \mathbb{N}$ did not converge to the number 0. Prove that $(a_n)$ does not converge (i.e., it diverges) by proving: For all $a \in \mathbb{R}$, $(a_n)$ does not converge to $a$. *Hint: You must consider various cases.*

2. Though we used the definite article "the" in saying *"we call a the limit of the sequence"*, nothing in the above explicitly rules out a sequence having more than one limit. Prove that this is not the case. That is, suppose that $\lim a_n = a$ and $\lim a_n = a'$ and prove that $a = a'$. *Hint: First, you can use the triangle inequality to show that* $|a - a'| \leq |a - a_n| + |a_n - a'|$. *Now make things small and use your result from the first exercise.*

3. In the spirit of the argument you gave in the previous proof, please prove the following: Let $(a_n)$ be a convergent sequence for which $a_n \leq M$ for all $n \in \mathbb{N}$, then $a = \lim_n a_n \leq M$.

   (a) Is it possible to weaken the hypotheses to say that $a_n \leq M$ for all but a finite number of $n$? Will the result still hold? Can you weaken it further?

   (b) Consider the analagous statement: If $(a_n)$ is convergent and $a_n < M$ for all $n \in \mathbb{N}$, then $a < M$. If this is true, prove it. If it is false, provide a counterexample.

**Exercise 4.** In the second edition of Abbott, please do:

1. Exercise 2.3.1
2. Exercise 2.3.3
3. Exercise 2.3.7
4. Exercise 2.3.9