Math 135: Homework 4 – not to be turned in

Below are four exercises that I would like you to do before Wednesday's midterm exam. As there won't be time to grade them and give you solution feedback, I will make solutions available to you on Monday. To reward you for doing these exercises before you receive solutions, please come to class on Monday with your solutions/attempts, show them to me, and I will give you some credit. In the meantime, if you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1 (The Binomial Theorem)**.** In this exercise, you prove the binomial theorem.

1. Show that, for any $1 \leq k \leq n-1$,
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$
Illustrate how this identity is the basic idea behind Pascal's triangle.

2. Use induction to prove the following theorem:

Theorem A (The Binomial Theorem)**.** *For any natural number n and real numbers x and y, we have*
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Exercise 2 (Infimum – the greatest lower bound)**.** In Abbott, please do:

1. Exercise 1.3.2:
   (a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or greatest lower bound of a set.
   (b) Now, state and prove a version of Lemma 1.3.7 for the greatest lower bound/infimum.
2. Exercise 1.3.3:
   (a) Let A be bounded below and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.
   (b) Use (a) to explain why there is no need to assert that the greatest lower bounds exists as part of the Axiom of Completeness.
   (c) Propose another way to use the Axiom of completeness to prove that sets bounds below have greatest lower bounds.

Exercise 3 (Properties of the supremum)**.** In Abbott, please do:

1. Exercise 1.3.4: Assume that A and B are nonempty subsets of real numbers which are bounded above and satisfy $A \subseteq B$. Prove that $\sup(A) \leq \sup(B)$.
2. Exercise 1.3.5: Let $A \subseteq \mathbb{R}$ be bounded above, and let $c \in \mathbb{R}$. Define the sets:
$$c + A := \{c + a : a \in A\} \qquad \text{and} \qquad cA = \{ca : a \in A\}.$$
   (a) Show that $\sup(c + A) = c + \sup(A)$.
   (b) If $c \geq 0$, show that $\sup(cA) = c \sup(A)$.
   (c) Postulate a similar type of statement for $sup(cA)$ for the case $c < 0$.
3. Exercise 1.3.6: Compute, without proofs, the suprema and infima of the following sets:
   (a) $\{n \in \mathbb{N} : n^2 < 10\}$
   (b) $\{n/(n+m) : n, m \in \mathbb{N}\}$

(c) $\{n/(2n+1) : n \in \mathbb{N}\}$

(d) $\{n/m : n, m \in \mathbb{N}\}$.

Exercise 4 (Consequences of Completeness)**.** In Abbott, please do:

1. Exercise 1.4.4: Use the Archimedean Property of $\mathbb{R}$ to rigorously prove that $\inf\{1/n : n \in \mathbb{N}\}$.

2. Exercise 1.4.5: Prove that $\cap_{n=1}^{\infty}(0, 1/n) = \varnothing$.

3. Exercise 1.4.6:Please do the following

   (a) Finish the proof of Theorem 1.4.5 by showing that the assumption that $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup(T)$.

   (b) Modify this argument to prove the existence of $\sqrt{b}$ for any non-negative real number b.