Math 135: Homework 3

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, September 26th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1 (Addition in Natural Number Systems)**.** In Section 2.4.1 of "Introduction to Mathematics: Number, Space, and Structure" by Scott Taylor, please do:

1. Exercise 2.38
2. Exercise 2.42

Exercise 2 (Natural numbers systems are all "the same")**.** In class, we discussed that $(\mathbb{N}, 0, n \mapsto n+1)$ is a natural number system (in that it satisfies the Peano axioms, Definition 2.27); here, we will take $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $(\mathbf{N}, \mathbf{0}, S)$ be another natural number system and define $f : \mathbb{N} \to \mathbf{N}$ by setting $f(0) = \mathbf{0}$, $f(1) = S(\mathbf{0})$, and, in general,

$$f(n+1) = S(f(n))$$

for $n \geq 0$. Please do the following:

1. Argue that, for every $n \in \mathbb{N}$ with $n \geq 1$, f is simply the n-fold composition of S evaluated at $\mathbf{0}$. In other words, argue that
$$f(n) = (\underbrace{S \circ S \circ S \circ S \circ \cdots \circ S}_{n})(\mathbf{0}) = \underbrace{S(S(S(\cdots(S(S}_{n}(\mathbf{0}))))$$
for $n \geq 1$.
2. Prove that f is surjective (onto). Hint: You can show that the range of f is a counting subset of $\mathbf{N}$ and invoke $(P3)$.
3. Prove that f is injective (one-to-one). Hint: Make use of $(P2)$ which, as we discussed in class, guarantees the injectivity of S.
4. Conclude that f is an bijection and therefore $\mathbb{N}$ and $\mathbf{N}$ are in one-to-one correspondence[1]

Exercise 3 (Proof Practice)**.** Please do

1. Exercise (1) in Section 4.8 of "Introduction to Mathematics: Number, Space, and Structure" by Scott Taylor.
2. Exercise (5) in Section 4.8 of "Introduction to Mathematics: Number, Space, and Structure" by Scott Taylor.

Exercise 4 (De Morgan's Laws (for sets))**.** Please do the following:

1. Do Exercise 1.2.3 of Abbott (Section 1.2)
2. Do Exercise 1.2.12 of Abbott (Section 1.2)

[1] This is part of the proof that all natural number systems are essentially the same. Interpreted otherwise: there is essentially only one natural number system as any other is in one-to-one correspondence with $\mathbb{N}$. A full proof of the result must also go beyond showing that f is a bijection and show that it also "plays well" with arithmetic; the first such proof was given by Richard Dedekind in 1888.