Math 135: Homework 2

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus[1]. Your solutions are due on Thursday, September 19th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1 (Function Composition)**.** Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$ be functions. We can define a new function called the composition of g and f written $g \circ f : A \to C$ and defined by

$$(g \circ f)(a) = g(f(a))$$

for each $a \in A$. This is the function that first carries a to $b = f(a)$ by f and then carries $b = f(a)$ to $c = g(b) = g(f(a))$ by g.

1. Give an example of sets A, B, C and functions $f : A \to B$ and $g : B \to C$ and produce the composition $g \circ f$. Does $f \circ g$ make sense? Explain.
2. Under what conditions (say, on A, B, C, f, g,) will $g \circ f$ and $f \circ g$ both make sense?
3. Show that, if f and g are both surjective, then $g \circ f$ is also surjective. *Hint: You must directly use the definition of onto (from the previous homework) to do this. Start with, let $c \in C$ and .... Eventually, you want to find an $a \in A$ that gets mapped to c under $g \circ f$.*
4. Show that, if f and g are both injective, then $g \circ f$ is also injective. *Hint: It's helpful here to use the characterization of one-to-oneness that says a function $h : A \to B$ is injective when the equation $h(a_1) = h(a_2)$ can only hold provided that $a_1 = a_2$.*
5. Use what you just showed above to argue that the composition of two bijections is a bijection.

Exercise 2. Let A and B be sets and suppose that $f : A \to B$. If f is a bijection, we can form a new function called f *inverse* (written $f^{-1} : B \to A$ in the following way: For each $b \in B$, there is some $a \in A$ such that $f(a) = b$ since f is onto. Also, this a is unique (it is the only such a) for which $f(a) = b$ because f is one-to-one. So, with this we set $f^{-1}(b) = a$ whenever a is the unique element in A for which $f(a) = b$. Please do the following.

1. Give an example of two finite sets A and B (each with 4 elements) and a bijection f between them.
2. Write down the function f^{-1} (you can express it either as a table or simply by giving all values).
3. Compute the composition $(f^{-1} \circ f)(a)$ for each element a of A.
4. Compute the composition $(f \circ f^{-1})(b)$ for each element b of B.
5. Now, work in generality: Let $f : A \to B$ be a bijection. Argue that what you found in the previous two items (in your specific example) is always true. In other words, what must $(f^{-1} \circ f)(a)$ be always and why? Also, what must $(f \circ f^{-1})(b)$ be always and why.

Exercise 3. Recall, that we say two sets X and Y are in one-to-one correspondence if there is a bijection $f : X \to Y$. Please do the following:

1. Show that every (non-empty) set A is in one-to-one correspondence with itself.
2. Show that, if for sets A and B, there is a bijection $f : A \to B$ then there also must be a bijection $g : B \to A$. This shows that the property of two sets being in one-to-one correspondence is **symmetric**.
3. Show that, if A is in one-to-one correspondence with B and B is in one-to-one correspondence with C then A is in one-to-one correspondence with C. This is called **transitivity**.

[1] Now is an excellent time, if you haven't already, to read the syllabus carefully.

Exercise 4 (Sets and Operations)**.** Let A and B be sets. We say that A **is a subset of** B and write $A \subseteq B$ if every element of A is also an element of B. In symbols, this means that, for every $x \in A$, $x \in B$. We say that A and B are equal if $A \subseteq B$ and $B \subseteq A$. In other words, A and B have exactly the same elements which is to say that $x \in A$ if and only if $x \in B$.

Consider the sets

$$S_0 = \{x \in \mathbb{Q} : x > 0\}, \qquad S_1 = \{x \in \mathbb{Q} : x^2 > 1\}, \qquad \text{and} \qquad S_2 = \{x \in \mathbb{Q} : x < -1\}.$$

1. Determine if $S_k \subseteq S_j$ for $k, j = 0, 1, 2$. In other words, investigate all of the set inclusions between S_0, S_1, and S_2.

Given two set A and B, we can form new sets in the following way:

- The intersection of A and B is the set

$$A \cap B = \{x : x \text{ is a member of both } A \text{ and } B\}.$$

  In other words, the intersection $A \cap B$ consists of all of those elements which belong to both A and B.

- The union of A and B is the set

$$A \cup B = \{x : x \in A \,\text{or}\, x \in B\}.$$

When talking about sets, there is one set that will keep coming up. It is called the empty set and often denoted by $\varnothing$. The empty set is, by definition, the set which has no elements.

2. Determine all possible (pairwise) unions of S_0, S_1, and S_2.
3. Show that, for any sets A and B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
4. Show that, for any sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
5. When is it the case that $A = A \cup B$? Justify your assertion.
6. When is it the case that $A = A \cap B$? Justify your assertion.

Note: To show that inclusion $\subseteq$ holds, take an arbitrary element of the set on the left and use your knowledge of that element (i.e., what does membership mean) to show it is in the set to the right.

Exercise 5. Please do Exercises 3.7 and 3.8 on Page 53 of "Introduction to Mathematics: Number, Space, Structure." by Scott Taylor.

Exercise 6. Please do Exercises 3.21 and 3.27 in "Introduction to Mathematics: Number, Space, Structure." by Scott Taylor.

Exercise 7. In Section 3.6 in "Introduction to Mathematics: Number, Space, Structure" by Scott Taylor, please do Exercises (1), (3), and (4).

Exercise 8. In Section 3.6 in "Introduction to Mathematics: Number, Space, Structure" by Scott Taylor, please do Exercises (8) Parts (a), (b), (d), and (m).