Chapter 1

The Real Number System

1.0 Introduction

This chapter introduces the basic mathematical object that underlies all of analysis: the real number system. No doubt you are already familiar with real numbers from your previous study of calculus, at least in an intuitive working sense. Here we will explore the real numbers more deeply. Our study begins with the Greeks' discovery of irrational quantities. The early Greek conception of number was based on whole numbers and quantities derived directly as the ratio of whole numbers. Since irrational quantities cannot be described in this concrete way, they were not regarded as proper numbers by the Greeks. For a long time irrationals, as their name indicates, occupied an anomalous status as phantom quantities scattered among the true "rational" numbers. It was not until the nineteenth century that a precise and logically sound meaning for the irrationals as numbers was developed by *constructing* the set of real numbers (rationals and irrationals) directly from the rational numbers.

In this construction the real numbers are endowed with a special property called completeness, which is the basis for all limit processes in calculus. In Section 1.2 we explore this construction and some of the consequences of this special completeness property. Following this, we examine the set of axioms that underlie the real number system and see how the common operations with real numbers are consequences of these axioms. Finally, in Section 1.4 the completeness of the real numbers is used to prove the important *Heine-Borel theorem*, which forms the foundation for several key results in the theory of calculus.

Mathematics evolves by abstracting and generalizing the properties and structures that it uncovers in the course of investigating particular mathematical objects. Perhaps no other object has inspired more mathematics in this way than the real number system. Each of the major branches of mathematics can trace its roots back to the fundamental example of the real numbers.

In **analysis** the completeness property of real numbers and the attendant notions of limit are the primary motivating ideas. In **algebra** the stucture of addition and multiplication of real numbers forms the central paradigm. The fundamental concept of *orders of infinity* in **logic** arose from questions about the size of the set of real numbers. Finally, the branch of **topology** originated in the study of open and closed sets of real numbers whose significance was highlighted by the Heine-Borel theorem. Thus, this single object, the real number system, has been the primary inspiration and guiding hand in the development of a large part of mathematics. A thorough study of the real numbers is indispensable for all serious mathematics students.

Foundational problems such as providing clear and precise definitions and rules for operation are often misunderstood as merely pedantic exercises. When first encountering such material we may tend to rely on our original intuitions of the subject, ignoring the difficult definitions and constructions. But good definitions need to be taken seriously, for they are actually meant to reshape our intuitions. It is with these newly reshaped tools that we are able to resolve the subtle confusions in our earlier understanding. Furthermore, these definitions and constructions form the foundation for exploration in areas where our earlier notions would have had no meaning.

At first, the new definitions and constructions seem abstract and contrived. However, with work they become more concrete and new, very keen intuitions develop. At its best, a thorough foundational study can be a truly enlightening experience. It can profoundly simplify and unify a subject, opening the way to powerful generalizations. In mathematics there is perhaps no better or more important foundational study than that of the real number system.

1.1 Irrational Numbers

The Greeks were the first civilization to view mathematics as an abstract deductive system. Earlier cultures had considered mathematics primarily as a tool kit of techniques to aid practical computations in activities such as navigation, construction, and commerce. The Greeks' interest in mathematics went beyond these practical applications to a deep concern for the logical integrity and consistency of arguments. This interest was rooted in a deep, almost religious belief in the power of pure rational thought to enlighten the mind. By training the mind logically, particularly through the study of mathematics, mortals could glimpse the underlying rational design of the universe and learn to lead a moral and just life. The following words of Proclus, a chronicler of early Greek mathematics, epitomize the lofty role of mathematics in Greek culture.

> This, therefore is mathematics; she reminds you of the invisible form of the soul; she gives life to her own discoveries; she awakens

> the mind and purifies the intellect; she brings light to our intrinsic ideas; she abolishes oblivion and ignorance which are ours by birth.[1]

The Pythagoreans

The elevation of mathematics to this preeminent status was the work of a famous school of philosophers known as the Pythagoreans, which arose in Greece about the sixth century BCE under the leadership of the mathematician Pythagoras. They lived as a secretive communal society, sharing their worldly goods and following strict dietary codes. As scholars the Pythagoreans were chiefly concerned with their studies, which they organized into four main branches of learning: *arithmetica* (number theory), *harmonia* (music), *geometria* (geometry), and *astrologia* (astronomy).

The Pythagoreans were motivated by the belief that through pure rational thought they could uncover and understand the design of the universe and how it worked. A basic tenet of this belief system was that numbers, by which they meant the positive whole numbers or the **natural numbers**, formed the key to understanding in all fields of knowledge. (We will use $\mathbb{N}$ to denote the natural numbers and $\mathbb{Z}$ to denote the set of all integers.) In the words of a famous follower, Philolaus:

> All things which can be known have number; for it is not possible that without number anything can be either conceived or known.[2]

The central role assigned to numbers by the Pythagoreans was mingled with mystical tendencies and led to a great deal of numerology. It was believed that objects could be represented by numbers and that the relationships between the numbers revealed truths about the relationships between the corresponding objects.

> The number one is the generator of numbers, the number of reason; two is the first even or female number, the number of opinion; three is the first true male number, the number of harmony, being composed of unity and diversity; four is the number of justice or retribution, indicating the squaring of accounts; five is the number of marriage, the union of male and female numbers; and six is the number of creation.[3]

For the Pythagoreans ratios of integers were the most important tool in the study of these relationships and, thus, formed the unifying thread connecting

---

[1] Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), p. 24.

[2] Carl B. Boyer, *A History of Mathematics* (Princeton, N.J.: Princeton University Press, 1985), p. 60.

[3] Boyer, p. 57.

all the areas of learning. In music, fundamental integral ratios underlying the theory of harmony were studied. These ideas were applied to *astrologia* in which it was believed that the motions of heavenly bodies were governed by the same ratios, and students ostensibly learned to hear "the music of the spheres."

In the same way it was believed that the basic truths of geometry could be revealed and understood through numbers and their ratios. A major tenet of this system was that any two line segments were **commensurable** (evenly measurable using a common unit). That is, for any two line segments it was believed possible to find a unit small enough so that each segment would be an exact integral multiple of that unit.

Figure 1.1.1. Two commensurable line segments.

Ironically, the Pythagoreans, themselves, discovered the fallacy in this belief. Certain familiar lengths such as the side of a square and its diagonal are inherently **incommensurable** and, hence, cannot be evenly measured by a common unit no matter how small. The existence of incommensurable lengths was a mysterious and incomprehensible fact to the Pythagoreans. Since the ratios between such lengths could not be described by whole numbers, they were called *alogos* meaning "without word" or inexpressible. This discovery implied that whole numbers and their ratios were in some way inadequate or too incomplete to describe geometric lengths. Obviously this was a major setback for a philosophy that had postulated number as the central key to all understanding. The problem was so embarrassing that according to legend the discoverer was thrown overboard at sea and members were forbidden to reveal the secret of the inexpressible quantities.[4]

Today we may find some of the Pythagoreans' beliefs naïve and superstitious, but we owe a great deal to these philosophers. They inspired some of the basic principles of western culture: that nature is susceptible to systematic rational understanding, and that moral conduct and justice should rest on logical reasoning. The four areas of learning that the Pythagoreans outlined were the original four "liberal arts" known as the *Quadrivium*. In the Middle Ages three more subjects, logic, rhetoric, and grammar, called the *Trivium*, were added. In the West these seven liberal arts have for centuries been considered the foundation of all learning. Notice the central role of mathematics in these liberal studies. Even the word *mathematics*, which means "that which is learned", is attributed to Pythagoras.

[4]Ernest Sondheimer and Alan Rogerson, *Numbers and Infinity: A Historical Account of Mathematical Concepts* (Cambridge: Cambridge University Press, 1981), p. 43.

Irrational Numbers

Let's examine how the problem of incommensurable lengths arose. Suppose for a moment that the side of a square and its diagonal *were* commensurable (Figure 1.1.2). This would mean there exists a unit small enough so that the side would be some integer multiple of this unit, say N, and the diagonal would be another whole multiple of the same unit, say M.

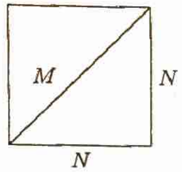

Figure 1.1.2. Can the diagonal and side of a square be commensurable?

By the Pythagorean theorem these numbers must satisfy $N^2 + N^2 = M^2$. That is, $2N^2 = M^2$ or

$$2 = \frac{M^2}{N^2}.$$

This last equality would then imply that $\sqrt{2} = M/N$ for integers M and N. More specifically, any number that can be expressed as such a quotient of two integers with $N \neq 0$ is called a **rational number**. We will denote the set of all rational numbers by $\mathbb{Q}$. Using the Pythagorean theorem we see that if the diagonal and side of a square are commensurable, then $\sqrt{2}$ must be rational. But the following argument shows that $\sqrt{2}$ is **irrational**, that is $\sqrt{2}$ cannot be expressed as the ratio of two integers.

Theorem 1.1.1. *There are no integers M and N such that $\sqrt{2} = M/N$.*

PROOF (By contradiction.) Suppose that $\sqrt{2} = M/N$ for some integers M and N. Since any fraction can be expressed in lowest terms by dividing out common factors, *we may further assume that M and N have no common divisors.* We will obtain a contradiction by showing that M and N must both be divisible by 2.

If $\sqrt{2} = M/N$, we may write $2 = M^2/N^2$ and so

$$2N^2 = M^2.$$

Since N^2 is an integer, this implies that M^2 is divisible by 2 (i.e., that it is an even integer). What about M? If M were odd, M^2 would also be odd; thus, M must be even. We conclude that *if M^2 is divisible by 2, M must be divisible by 2.* So we may write $M = 2P$ for some integer P and substitute for M in the foregoing equation:

$$2N^2 = (2P)^2$$

or, equivalently,

$$N^2 = 2P^2.$$

The last equality says that N^2 is divisible by 2; hence, N is also divisible by 2. We have shown that M and N *have a common factor*, 2, contrary to our original choice of M and N. Therefore, there can be no integers M and N with $\sqrt{2} = M/N$. ■

The Greeks were masters at discovering inconsistencies buried in statements, as is beautifully illustrated in the dialogues of Socrates. They honed the technique of proof by contradiction, known as *reductio ad absurdum*, to a fine art. The preceding argument shows that if a rational expression for $\sqrt{2}$ existed, it could not have been in lowest terms. Since any ratio of integers can always be reduced to lowest terms, this must mean that no such expression can exist. It is considered one of the most beautiful examples of proof by contradiction in elementary mathematics. In the remainder of this section we look at several other famous proofs by contradiction.

The Fundamental Theorem of Arithmetic