Countability

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1 Instructions

The following instructions pertain to this and other handouts created by Leo. When attempting a TYC or an Exercise, you may use any of the results appearing prior to such (including in other TYC's and Exercises), unless it is specifically indicated that a result is a glimpse into the future (i.e. stated to indicate further developments).

2 Basic Terminology

2.1 Definition

The "$\mathbb{N}$otel" motel has infinitely many rooms, and these rooms are numbered by the natural numbers (i.e. positive integers). The room numbers are placed over the rooms. Here is how we can represent this $\mathbb{N}$otel:

No room has ∞ as its number!

1	2	3	4	5	6	7	8	9	10	11	12	13	...
													...

2.2 Terminology

When we talk about a collection of objects, it is understood that all of the objects in the collection are distinct.

We shall not bother with "empty" collections here.

- A collection of (distinct) objects can be **accommodated** in the $\mathbb{N}$otel if *all* of the objects can be placed into the rooms of the $\mathbb{N}$otel, with no two objects sharing a room.

- The $\mathbb{N}$otel is **of just the right size (JR)** for a given collection of objects if the collection can be *accommodated* in the ℕotel occupying every room (with no double occupancy!). We will express this by saying that the collection is **ℕ-sized**.

- A block of rooms starting with room number 1 and ending with room number 37 is said to be **the** 37**th initial block** of the ℕotel. Other initial blocks are defined similarly. Each initial block has a first and a last room in it, with the first room always being room number 1.

- A block consisting of all rooms starting with room number 38 is said to be **the** 38**th tail block** of the ℕotel. Other tail blocks are defined similarly. Each tail block has a first room, but no last room, in it.

- The concepts such as the 345th tail block of the ℕotel being JR for a given collection of objects, are now self-explanatory. Arguing that, say, the 86th initial block of the ℕotel is JR for a given collection amounts to claiming that there are 86 objects in the collection.

- **Finite** collections are those that can be accommodated in some initial block of the ℕotel.

- Collections that are not finite, are said to be **infinite**. *We shall take it for granted that every* ℕ*-sized collection is infinite.*
  We are NOT claiming that every infinite collection is ℕ-sized!

2.3 Example

For example, any collection of 435 objects can be accommodated in the ℕotel.

The set of all natural numbers greater than 3 is ℕ-sized:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	...
4	5	6	7	8	9	10	11	12	13	14	15	16	...

2.4 Test Your Comprehension ☞ ***Smaller collections are easier to accommodate***

Argue that if a given collection can be accommodated in the $\mathbb{N}$otel, and some (but not all) of the objects are removed from the collection, then the new collection can also be accommodated in the $\mathbb{N}$otel.

2.5 Test Your Comprehension ☞ ***There are just as many even positive integers as there are positive integers***

Argue that the collection $2, 4, 6, 8, \ldots$ of all even natural numbers is $\mathbb{N}$-sized.

3 Take it for granted.

An Axiom: The Well-Ordering Principle (WOP) of $\mathbb{N}$

In every non-empty collection of natural numbers, one of the numbers is smallest.

Note that the integers and the rational numbers do not have such a property.

3.1 Fact ☞ ***... for the time being, perhaps.***

Any non-empty finite collection will fill out exactly one initial block of the $\mathbb{N}$otel, and that is how we identify the size of the collection.

This "Fact" can be in fact derived as a theorem from the WOP, but doing so feels too pedantic for our purposes at the moment.

4 ℕ-sized collections

4.1 Test Your Comprehension ☞ ***A tail block has the same size as the whole ℕotel***

Argue that the following statements are true.

1. If some tail of the ℕotel is JR for a given collection, then the collection is actually ℕ-sized.

2. If a given collection is ℕ-sized, then every tail of the ℕotel is JR for the collection.

4.2 Test Your Comprehension ☞ ***Whats one more object, or a million, when you have a ℕotel!***

Argue that the following statements are true.

1. If a given collection is ℕ-sized, then the collection can also be accommodated in the ℕotel in such a way that there is a room left unoccupied.

2. If a given collection is ℕ-sized, and a new object is added (i.e. "thrown in") to the collection, then the resulting collection is still ℕ-sized.

3. If a given collection is ℕ-sized, and 10^6 new objects are added (i.e. "thrown in") to the collection, then the resulting collection is still ℕ-sized.

4.3 Test Your Comprehension ☞ ***Whats one less object, or a million, when you have a ℕotel!***

Argue that the following statements are true.

1. If a given collection is ℕ-sized, and an object is removed from the collection, then the resulting collection is still ℕ-sized.

2. If a given collection is ℕ-sized, and 10^6 objects are removed from the collection, then the resulting collection is still ℕ-sized. (What if we do not specify how many objects have been removed?)

4.4 Exercise

Argue that if a given collection is ℕ-sized, then the collection can also be accommodated in the ℕotel in such a way that there are infinitely many unoccupied rooms left.

4.5 Exercise ☞ ***Valentine's Day ℕotel I***

Argue that if a given collection is ℕ-sized, and each of the objects in the collection "brings a friend", then the new collection comprised of the original objects and their friends is also ℕ-sized.

4.6 Exercise ☞ ***ℕotel and ℤ***

Argue that each of the following collections is ℕ-sized.

1. The collection $\ldots, -3, -2, -1, 1, 2, 3, \ldots$ of all non-zero integers.

2. The collection $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ of all integers.

4.7 Theorem ☞ ***The ℕotel has the smallest infinite capacity***

If a given *infinite* collection can be accommodated in the ℕotel, then the collection is actually ℕ-sized.

Proof of Theorem 4.7. Suppose that a collection $\mathcal{C}$ has been accommodated in the ℕotel.

Each natural number has finitely many natural numbers smaller than it (may be none). For each occupied room number k, let n_k be the number of occupied rooms with numbers less than k. (This makes sense by Fact 3.1.) Place the occupant of the room number k into the room number $n_k + 1$. Do this simultaneously for all occupied rooms. Consider the new arrangement. Obviously no element of $\mathcal{C}$ is now "roomless." The following claims, the verification of which is left to the reader, complete the argument.

Claim 0: If a given $A \in \mathcal{C}$ occupied a room with a higher number than that of the room occupied by a given $B \in \mathcal{C}$ in the original arrangement, then the same is true in the new arrangement as well. (TYC 4.8)

Claim 1: In the new arrangement no room is occupied by more than one element of $\mathcal{C}$. (TYC 4.9)

Claim 2: If a given $A \in \mathcal{C}$ and a given $B \in \mathcal{C}$ were consequent occupants in the original arrangement (i.e. there were no occupied rooms between their rooms), then they are immediate neighbors in the new arrangement. (TYC 4.10)

Claim 3: There are no unoccupied rooms in the new arrangement. (TYC 4.11) ■

4.8 Test Your Comprehension

Verify Claim 0 in the proof of 4.7.

4.9 Test Your Comprehension

Verify Claim 1 in the proof of 4.7.

4.10 Test Your Comprehension

Verify Claim 2 in the proof of 4.7.

4.11 Test Your Comprehension

Verify Claim 3 in the proof of 4.7.

4.12 Test Your Comprehension

Argue that every infinite collection of natural numbers is $\mathbb{N}$-sized.

Hint: (for TYC 4.11) First argue that room number 1 is occupied. Now suppose some room number m is unoccupied. Obviously $m > 1$. Argue that there is an occupied room to the left of room number m, and - to the right. Argue that one of the occupied rooms to the left of the room number m has the largest room number (among all such). Let us say this room is occupied by A. Argue that one of the occupied rooms to the right of the room number m has the smallest room number (among all such). Let us say this room is occupied by B. Argue that in the original arrangement A and B were consequent occupants (i.e. there were no occupied rooms between their rooms). Conclude that in the new arrangement A and B are immediate neighbors, contradicting the emptiness of the room number m. Hence no such empty room can exist.

4.13 Exercise ☞ ***Valentine's Day $\mathbb{N}$otel II***

Argue that if a given collection is $\mathbb{N}$-sized, and *some* of the objects in the collection "bring a friend", then the new collection comprised of the original objects and their friends is also $\mathbb{N}$-sized.

4.14 Exercise ☞ ***$\mathbb{N}$otel mergers***

Argue that the following statements are true.

1. If each of two given collections is $\mathbb{N}$-sized, then so is their union. It may happen that some objects were common to the two collections. Only one copy of such an object appears in the union.

2. If each of three given collections is $\mathbb{N}$-sized, then so is their union.

5 $\mathbb{H}$otel

Suppose now that we have not just one but infinitely many distinct $\mathbb{N}$otels:

$$\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3, \ldots$$

which are numbered/indexed by the natural numbers, and we use these to construct an infinitely tall $\mathbb{H}$otel by making $\mathbb{N}_1$ be its ground floor, $\mathbb{N}_2$ – its second floor, $\mathbb{N}_3$ – its third floor, etc. So the $\mathbb{H}$otel looks something line this:

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
$\mathbb{N}_5$														...
$\mathbb{N}_4$														...
$\mathbb{N}_3$														...
$\mathbb{N}_2$														...
$\mathbb{N}_1$														...
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	...

where we have placed the room numbering at the bottom, since it is common to every floor.

Clearly we can talk about collections of objects that can be accommodated in the $\mathbb{H}$otel, or about the $\mathbb{H}$otel being JR for a given collection.

5.1 Test Your Comprehension ☞ *The $\mathbb{N}$otel has no more capacity than the $\mathbb{H}$otel*

Convince yourself that if a collection of objects can be accommodated in the $\mathbb{N}$otel, it can also be accommodated in the $\mathbb{H}$otel.

5.2 Theorem ☞ *Cantor's Snake: the $\mathbb{H}$otel has no more capacity than the $\mathbb{N}$otel*

If a collection can be accommodated in the $\mathbb{H}$otel, it can be accommodated in the $\mathbb{N}$otel.

Proof of Theorem 5.2. Outlined in a video. ■

The striking point here is that we cannot accommodate any more collections in the $\mathbb{H}$otel than we could in the $\mathbb{N}$otel. Of course, if it were true that ANY infinite collection can be accommodated in the $\mathbb{N}$otel, (i.e. that any infinite collection is $\mathbb{N}$-sized) then there would be no surprise in this. But is any infinite collection $\mathbb{N}$-sized?

5.3 Test Your Comprehension ☞ *$\mathbb{H}$otel and $\mathbb{N}$otel have the same capacity*

Argue that the following statements are true.

1. If the $\mathbb{H}$otel is JR for a given collection, then the collection is $\mathbb{N}$-sized.

2. If a given collection is $\mathbb{N}$-sized, then the $\mathbb{H}$otel is also JR for that collection.

5.4 Theorem ☞ *$\mathbb{H}$otel and $\mathbb{Q}^+$*

The collection of all positive rational numbers can be accommodated in the $\mathbb{H}$otel.

Proof of Theorem 5.4. Outlined in a video. ■

5.5 Test Your Comprehension ☞ *ℕotel and ℚ*

Argue that each of the following collections is ℕ-sized.

1. The collection of all positive rational numbers.
2. The collection of all non-zero rational numbers.
3. The collection of all rational numbers.

5.6 Theorem ☞ *Cantor's Diagonal*

The collection of all real numbers between 0 and 1 can<u>not</u> be accommodated in the ℕotel.

Proof of Theorem 5.6. Outlined in a video. ∎

5.7 Exercise ☞ *ℕotel and ℝ*

Argue that the following statements are true.

1. The collection of all real numbers can<u>not</u> be accommodated in the ℕotel.
2. The collection of all irrational numbers can<u>not</u> be accommodated in the ℕotel.