The Method of Characteristics

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In this primer, we outline the so-called method of characteristics for PDEs of the form
$$u_t + c(t,x,u)u_x = f(t,x,u)$$
where $c = c(t,x,u)$ is some real-valued function. Observe that the advection-decay equation and non-linear advection equation,
$$u_t + c(t,x)u_x = f(t,x,u)$$
and
$$u_t + c(u)u_x = 0,$$
respectively, are of this form (with $c(t,x,u) = c(t,x)$ for the first and $c(t,x,u) = c(u)$ for the second). The goal of the method of characteristics is to find a suitable coordinate change and apply the chain rule to reduce the given PDE into a family of ODEs which are easier to solve. The basic ingredients of this method are as follows:

1. Find a locally (hopefully globally) invertible coordinate transformation $(t,x) \overset{T}{\longmapsto} (\tau,\xi)$ for which $t = \tau$ and $T(0,x) = (0,\xi)$. We may therefore write $T(t,x) = (t,\xi(t,x))$ for some function $\xi = \xi(t,x)$ satisfying $\xi(0,x) = x$ for all x. Further, given that T is invertible, $T^{-1}(\tau,\xi) = (\tau, x(\tau,\xi))$ for some function $x = x(\tau,\xi)$ satisfying $x(0,\xi) = \xi$ for all ξ.
2. Define $U(\tau,\xi) = u \circ T^{-1}(\tau,\xi) = u(\tau, x(\tau,\xi))$ and, with it, obtain a family of ODEs in U equivalent to the given PDE.

In what follows, we use the two basic ingredients above to derive two coupled families of ODEs and their corresponding IVP which are equivalent to the initial value problem
$$\begin{cases} u_t + c(t,x,u)u_x = f(t,x,u) & \text{for } t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{1}$$
that we aim to solve; here, the initial "data" $u_0(x)$ is known. Solving this coupled systems of ODEs (see (2) and (3)) will furnish a solution to (1). We first deal with a technical lemma.

Lemma 1. *Suppose that $\xi = \xi(t,x)$ is once continuously differentiable on an open set $\mathcal{O} \subseteq \mathbb{R}^2$ and has $\partial\xi/\partial x \neq 0$ on $\mathcal{O}$. Then*

1. *The transformation T is once continuously differentiable (in the sense of vector-valued functions) on $\mathcal{O}$ and its range, $\mathcal{R} = T(\mathcal{O})$, is an open set.*

2. *The transformation T is (locally) invertible with continuously differentiable inverse $T^{-1}(\tau, \xi) = (\tau, x(\tau, \xi))$ for which $\partial x/\partial\xi = 1/(\partial\xi/\partial x)$ and $\partial x/\partial\tau = -\partial\xi/\partial t/(\partial\xi/\partial x)$.*

Under these conditions, we let $\mathcal{G} \subseteq \mathcal{R}$ be the domain of $T^{-1}(\tau, \xi) = (\tau, x(\tau, \xi))$ (and, in particular, $T^{-1}(\mathcal{G}) \subset \mathcal{O}$ and $T \circ T^{-1}$ is the identity map on $\mathcal{G}$). Then, if U is continuously differentiable on $\mathcal{R}$ and is such that $U(0, \xi) = u_0(\xi)$ for all ξ, then $u := U \circ T$, i.e., $u(t, x) = U(t, \xi(t, x))$ is continuously differentiable on $\mathcal{O}$,

$$U_\tau(\tau, \xi) = u_t(\tau, x(\tau, \xi)) + \frac{\partial x}{\partial \tau} u_x(\tau, x(\tau, \xi))$$

for all $(\tau, \xi) \in \mathcal{G}$ and

$$u(0, x) = u_0(x)$$

for all x.

Proof. Items 1 and 2 are consequences of the Inverse Function Theorem (see, e.g., Theorem 9.24 and 9.25 of [1]) and are beyond the scope of this course[1]. By Item 2 and the chain rule, we have

$$\begin{aligned} U_\tau(\tau, \xi) &= \frac{\partial}{\partial \tau} u \circ T^{-1}(\tau, \xi) \\ &= \frac{\partial}{\partial \tau} u(\tau, x(\tau, \xi)) \\ &= u_t(\tau, x(\tau, \xi)) + u_x(\tau, x(\tau, \xi)) \left(\frac{\partial x}{\partial \tau}(\tau, x(\tau, \xi)) \right) \\ &= u_t(\tau, x(\tau, \xi)) + \frac{\partial x}{\partial \tau} u_x(\tau, x(\tau, \xi)) \end{aligned}$$

for $(\tau, \xi) \in \mathcal{G}$, i.e., for all $(\tau, \xi) \in \mathcal{R}$ for which T^{-1} is defined. Finally, because $T(0, x) = (0, x)$, we have

$$u(0, x) = u \circ T(0, x) = U(0, x) = u_0(x)$$

for all x. □

With the above setup in mind, we will reduce (1) into a pair of coupled (family of) ODEs in the following way. Set $F(\tau, \xi, U) = f(\tau, x(\tau, \xi), U)$ and observe that, if $U(\tau, \xi)$ satisfies the differential equation

$$U_\tau(\tau, \xi) = F(\tau, \xi, U)$$

subject to the initial condition that $U(0, \xi) = u_0(\xi)$, then by virtue of the lemma, $u = U \circ T$ satisfies the differential equation

$$u_t(t, x) + c(t, x, u)u_x(t, x) = f(t, x, u)$$

[1]The IFT is wonderfule and incredibly powerful and you should learn about it!

on $\mathcal{G}$ and the initial condition $u(0,x) = u_0(x)$ provided that

$$\frac{\partial x}{\partial \tau}(\tau,\xi) = c(\tau, x(\tau,\xi), U(\tau,\xi))$$

for $(\tau,\xi) \in \mathcal{G}$. Since we have required $x(0,\xi) = \xi$, we may view the preceding equation subject to the initial value problem $x(0,\xi) = \xi$. All together, we obtain the pair of families of initial value problems

$$\begin{cases} U_\tau = F(\tau,\xi,U) & \text{for } \tau > 0, \xi \in \mathbb{R} \\ U(0,\xi) = u_0(\xi) & \text{for all } \xi \in \mathbb{R} \end{cases} \tag{2}$$

and

$$\begin{cases} \frac{\partial x}{\partial \tau} = c(\tau, x(\tau,\xi), U) & \text{for } \tau > 0, \xi \in \mathbb{R} \\ x(0,\xi) = \xi & \text{for all } \xi \in \mathbb{R} \end{cases} \tag{3}$$

Upon noting that (3) depends on U through $c(\tau, x(\tau,\xi), U)$ and (2) depends on x through $F(\tau,\xi,U) = f(\tau, x(\tau,\xi), U)$, the above (family of) differential equations are coupled. If solutions U and x can be found, then, in particular, we know the transformation $T^{-1}(\tau,\xi) = (\tau, x(\tau,\xi))$ and we can use it to compute $\xi = \xi(t,x)$, the second component of T. Then, by the lemma and the computations which led to (2) and (3), we are guaranteed that

$$u(t,x) = U(t, \xi(x,t))$$

solves the initial value problem (1). In summary, this method allowed us to reduce the IVP (1) to the pair of coupled IVPS (2) and (3) which are, hopefully, easier to solve.

Let's work an example.

Example 1. Consider the advection and decay problem

$$\begin{cases} u_t + x u_x = 2tu & \text{for } t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$$

where u_0 is known. In this case, we have $c(t,x,u) = x$ and $f(t,x,u) = 2tu$. The fact that $c(t,x,u)$ is only a function of x (and not u) makes the (3) "uncoupled" from (2) and so we can solve (3) first. In this setting, this is

$$\begin{cases} \frac{\partial x}{\partial \tau} = x(\tau,\xi) & \text{for } \tau > 0, \xi \in \mathbb{R} \\ x(0,\xi) = \xi & \text{for all } \xi \in \mathbb{R} \end{cases}.$$

This family ODEs are all linear (in x) and you will find (and you should try) that the (unique) solution is

$$x(\tau,\xi) = \xi e^\tau$$

which gives

$$T^{-1}(\tau,\xi) = (\tau, \xi e^\tau)$$

and so the "characteristic transformation" is

$$T(t,x) = (t, xe^{-t})$$

implying that, in particular, $\xi = \xi(t,x) = xe^{-t}$. Let's now focus on (2). In the present setting $F(\tau,\xi,U) = f(\tau, x(\tau,\xi), U) = 2\tau U$ and so (2) is

$$\begin{cases} U_\tau = 2\tau U & \text{for } \tau > 0,\, \xi \in \mathbb{R} \\ U(0,\xi) = u_0(\xi) & \text{for } \xi \in \mathbb{R}. \end{cases}$$

Given our ODE theory for linear equations, we recognize the unique solution to the above equation (which you should find yourself) is given by

$$U(\tau,\xi) = u_0(\xi)e^{\tau^2}$$

for $(\tau,\xi) \in \mathbb{R}^2$. Thus, our theory predicts that

$$u(t,x) = U(t,\xi(t,x)) = U(t, xe^{-t}) = u_0(xe^{-t})e^{t^2}$$

solves the given IVP. You should verify directly that this, in fact, works (in doing so, feel free to assume $u_0 \in C^1(\mathbb{R})$.

References

[1] Walter Rudin. *Principles of Mathematical Analysis, 3rd Ed.* McGraw-Hill, 1976.