The cane problem

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Please watch Veritasium's video entitled "5 Fun Physics Phenomena" at http://youtu.be/1Xp_imnO6WE. The video depicts a man initially supporting a cane horizontally with his two index fingers. The center of the mass of the cane sits at an arbitrary point between his two fingers. At $t = 0$, he begins to slide his fingers toward one another (at possibly varying rates); he stops when his fingers meet at time T. The phenomenon observed is that, at $t = T$, the center of mass of the cane sits directly on top of his fingers and hence the cane balances on his fingers. The following is my solution to "the cane problem"; Figure 1 illustrates the situation.

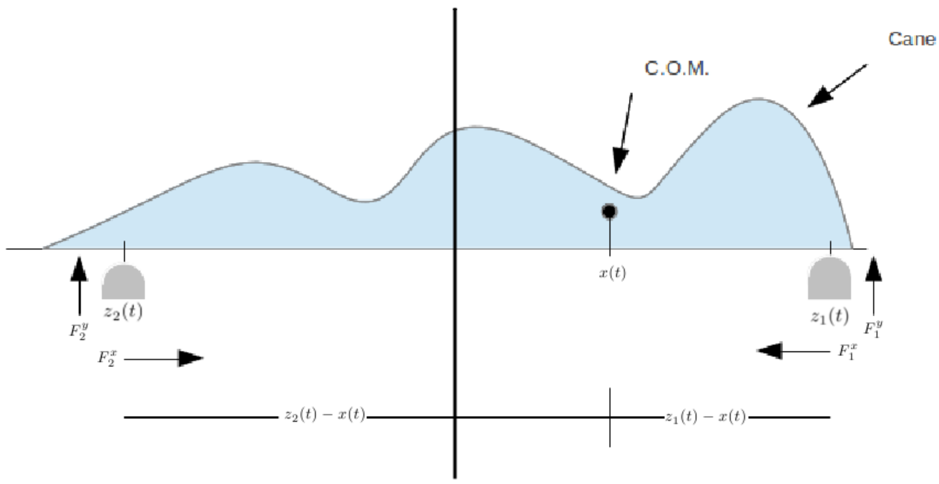

Figure 1: The model

Our frame of reference is the that of the person; we place the origin at the center of his chest. Let's first specify the motion of the fingers by $z_1, z_2 : [0, T] \to \mathbb{R}$ where we will make the following assumptions:

1. At time T, the fingers meet for the first time, i.e., $z_1(T) = z_2(T)$, but $z_1(t) \neq z_2(t)$ for all $t \in [0, T)$.

2. The fingers move in a reasonably smooth way, i.e., $z_1, z_2 \in C^2([0, T])$.

3. For all $t \in [0, T]$, the fingers move toward one another, i.e., $\dot{z_1}(t) \leq 0$ and $\dot{z_2}(t) \geq 0$ for all $t \in [0, T]$.

These assumptions are consistent with all situations depicted in the video. We note that item 3 isn't essential, it helps only with a sign convention. Now that we have specified the motion of the fingers, we will denote by $x : [0,T] \to \mathbb{R}$, the position of the center of the mass of the cane as a function of time. Our job is to find the equation of motion for x. Let's first consider the two static constraints, the vertical motion and the torque.

1 Statics

As the person moves his fingers in such a way that the cane remains horizontal, the weight of the cane must be balanced by the upward force of the fingers. If we denote these forces by $F_1^y, F_2^y : [0,T] \to \mathbb{R}$ respectively, we have

$$mg = F_1^y(t) + F_2^y(t)$$

for all $t \in [0,T]$, where m is the mass of the cane and g is the gravitational constant. We will also assume that the person keeps the cane horizontal on the interval $[0,T]$; this is done by continuously adjusting the forces F_1^y and F_2^y to balance the torque. Thus,

$$0 = F_1^y(t)(z_1(t) - x(t)) + F_2^y(t)(z_2(t) - x(t))$$

for all $t \in [0,T]$. Note that $|z_i(t) - x(t)|$ for $i = 1,2$ are precisely the lengths of the moment arms about the center of mass of the cane. Combining our equations we have

$$(z_1(t) - z_2(t))F_1^y(t) = mg(x(t) - z_2(t)) \tag{1}$$

and

$$(z_2(t) - z_1(t))F_2^y(t) = mg(x(t) - z_1(t)) \tag{2}$$

for all $t \in [0,T]$.

2 Friction

It seems reasonable to me that the horizontal force on the cane (by each finger) depends (directly) on the vertical force. The most naive thing that corresponds to my intuition is the following assumption:

- There exists $\alpha > 0$ for which $F_i^x(t) = \mathrm{sgn}(\dot{z}_i(t))\alpha F_i^y(t)$ for $i = 1,2$ and for all $t \in [0,T]$.

The constant α is a "*frictional*" coefficient and we note that it always forces the cane in the finger's direction of motion provided $F_i^y(t) \geq 0$ for $i = 1,2$ and for all $t \in [0,1]$. Note also that α is the same for each finger and is independent of position and time. By our assumptions on $\dot{z}_i$ for $i = 1,2$,

$$F_1^x(t) = -\alpha F_1^y(t) \;\text{ and }\; F_2^x(t) = \alpha F_2^y(t) \tag{3}$$

for $t \in [0,T]$.

3 The equation of motion

For the center of mass, our set-up has constrained all direction of motion except the horizontal direction. Appealing to (3) and Newton's second law,

$$m\ddot{x}(t) = F_1^x(t) + F_2^x(t) = \alpha(F_2^y(t) - F_1^y(t))$$

for $t \in [0, T]$. By virtue of (1) and (2), we have

$$(z_1(t) - z_2(t))\ddot{x}(t) = \alpha g[(z_1(t) + z_2(t)) - 2x(t)]$$

and hence, for all $t \in [0, T]$,

$$x(t) = \frac{1}{2}(z_1(t) + z_2(t)) - \frac{1}{2\alpha g}(z_1(t) - z_2(t))\ddot{x}(t). \tag{4}$$

Theorem 3.1. $x(T) = \frac{1}{2}(z_1(T) + z_2(T))$. *Thus, if our assumptions concerning the frictional forces correspond to reality, when the fingers meet at* $t = T$*, the center of mass of the cane is directly above the fingers.*

Proof. Given our assumptions concerning z_1 and z_2, the coefficients of the differential equation (4) are sufficiently regular to ensure that (4) has a solution $x \in C^2([0, T])$; this solution is unique upon specifying initial values $x(0)$ and $\dot{x}(0)$. For each such solution x, it is evident that $x(T) = (z_1(T) + z_2(T))/2$ in view of (4). □

Let's note that nothing in our construction used the assumption that $z_2(0) \leq x(0) \leq z_1(0)$ which was the situation in the video. If this initial constraint does not hold, it isn't physically reasonable to expect that $x(T) = (z_1(T) + z_2(T))/2$. So why does the Theorem still hold in this case? Answer: It's the shortcoming of our model. In the case that say, $z_2(0) \leq z_1(0) \leq x(0)$ we should expect the finger at $z_2(0)$ to produce a downward force to balance the torque, i.e., the finger would have to be on top of the cane. In this case however, our assumptions concerning friction is completely nonsensical; the model predicts that the frictional force due to the finger at $z_2(0)$ works in the negative x direction.