Before beginning this homework assignment, please review the guidelines for submitting homework. Remember to write down the total amount of time spent working on the assignment at the top of what you turn in. These problems give you some exposure to the proper organization and structure of a well-written proof. They also force you to review previous material and help you to notice how some very basic properties of numbers still require proofs that we cannot yet provide.

(1) (G) In this problem you will prove that $\sqrt{3}$ is an irrational number. You will need the following definition:

**Definition.** A number $x$ is **rational** if there exist integers $a$ and $b \neq 0$ such that $x = \frac{a}{b}$. A number is **irrational** if it is not rational.

Prove that $\sqrt{3}$ is irrational. When you write up your solution, you should list all elementary number facts that you need (some of which you may not be able to prove.) Any portion of the proof that can be separated out as a separate lemma should be.

(This is your chance to adapt the examples from class to prove something rather sophisticated. You should focus on writing something that is coherently organized and logically correct. At this point you may have gaps in your argument or statements you don’t know how to justify. These should be clearly marked as such. You should base your work on the proof done in class that $\sqrt{2}$ is irrational. Rather than discussing even numbers, you will need to discuss numbers that are multiples of 3. Also, this is certainly a proof that you can find online – but you shouldn’t do that! Instead work hard to construct it for yourself.)

(2) (G) Prove that there exists a function $f : [0, 1] \to \mathbb{R}$ that is differentiable on $(0, 1)$ such that there does not exist $x \in (0, 1)$ with

$$f'(x) = f(1) - f(0)$$

(The notation $f : [0, 1] \to \mathbb{R}$ means that any number from the interval $[0, 1]$ can be put into $f$ and then $f$ will spit out a real number – so this is the kind of function that you’ve often encountered in high school algebra or calculus.)
Hint: the function cannot be continuous at $t = 0$ or $t = 1$ since otherwise the mean value theorem would imply that such an $x$ does exist.)

(3) (G) Suppose that $f: [0, 1] \rightarrow [0, 1]$ is a function such that for all $t \in [0, 1]$

$$\frac{1}{4} t \leq f(t) \leq \frac{1}{2} t.$$ 

Prove that there exists a unique $x \in [0, 1]$ such that $f(x) = x$. (Recall that you must show that such an $x$ exists and that it is unique.)

(4) (G) Prove (using a proof by contradiction and some algebra) that the curves in $\mathbb{R}^2$ defined by the equations

$$x^2 + 3xy + y^2 = 1$$

and

$$y = -2x$$

do not intersect.