Invariance of Dimension

The purpose of this lecture is to prove:

Theorem 1. If $n \neq m$, then no open set in $\mathbb{R}^n$ is homeomorphic to an open set in $\mathbb{R}^m$.

To prove this we will use Sperner's Lemma to prove that the topological dimension of an n–simplex Δ^n is equal to n.

Definition 2. Suppose that Δ^k is an n–simplex which is the convex hull of the affinely independent points $v_0, \ldots, v_n \in \mathbb{R}^n$. The **barycenter** of Δ^k is the point $(v_0 + \ldots + v_k)/(k+1)$.

Suppose that Δ^n is an n–simplex. The **first barycentric subdivision** $\mathscr{T}_1$ of Δ^n is the triangulation of Δ^n constructed in the following way. The vertices $\mathscr{T}^{(0)}$ of $\mathscr{T}$ are the points which are the barycenters of all the faces of Δ^n. Now we define the n–simplices of $\mathscr{T}$: $(k+1)$ affinely independent vertices $w_0, \ldots, w_k$ of $\mathscr{T}$ determine a n–simplex of $\mathscr{T}$ if and only if the convex hull of $\{w_0, \ldots, w_k\}$ intersects $\mathscr{T}^{(0)}$ only in the points $\{w_0, \ldots, w_k\}$. The ***p*th barycentric subdivision** $\mathscr{T}_p$ of Δ^n is obtained by taking the first barycentric subdivision of each of the n-simplices in $\mathscr{T}_{p-1}$. Notice that as $p \to \infty$, the diameter of each n–simplex in pth barycentric subdivision is converging to zero.

Theorem 3. The topological dimension of Δ^n is at least n.

Proof. By definition, the dimension of Δ^n is less than or equal to K if, for all $\varepsilon > 0$, there exists a finite closed cover of Δ^n by sets of diameter less than ε which has order $K+1$. We wish to show that the dimension of Δ^n is not less than or equal $n-1$. Thus, we must show that there exists $\varepsilon > 0$ such that if $\mathscr{U}$ is a finite closed cover of Δ^n by sets of diameter less than ε, then the order of $\mathscr{U}$ is at least $n+1$.

Let $F_0, \ldots, F_n$ be the $n-1$ dimensional faces of Δ^n. Let a_i be the vertex of Δ^n which is opposite F_i. For each i, the set $\Delta^n \backslash F_i$ is open in Δ^n. The collection $\{\Delta^n \backslash F_i\}$ is an open cover of Δ^n. Let ε be the Lebesgue number of this cover. Suppose that $\mathscr{U}$ is a finite closed cover of Δ^n by sets of diameter less than ε. We wish to show that the order of $\mathscr{U}$ is at least $n+1$.

For each $U \in \mathscr{U}$, $\operatorname{diam}(U) < \varepsilon$. Since ε is the Lebesgue number of the open cover $\{\Delta^n \backslash F_i\}$, there exists i so that $U \subset \Delta^n \backslash F_i$. Define a function $\phi : \mathscr{U} \to \{0, \ldots, n\}$ so that $\phi(U) = i \Rightarrow U \subset \Delta^n \backslash F_i$.

Notice the following: If $\phi(U) = i$ then U is disjoint from F_i. Also, if $\phi(U) = i$ then either U is disjoint from the vertices $\{a_0, \ldots, a_n\}$ of Δ^n or a_i is the sole vertex of Δ^n which is contained in U. Since $\mathscr{U}$ is a cover of Δ^n, each vertex a_i is contained in some U for which $\phi(U) = i$.

Define
$$A_i = \bigcup_{\phi(U)=i} U.$$
Notice that
$$\bigcup_{i=0}^{n} A_i = \bigcup_{U \in \mathscr{U}} U = \Delta^n.$$
Notice also that $a_i \in A_i$ and that $A_i \cap F_i = \varnothing$.

We now set out to use Sperner's Lemma:

Let $\mathscr{T}_p$ be the pth barycentric subdivision of Δ. To each $x \in \Delta$ assign a label $L(x) = \min\{i | x \in A_i\}$. Notice that $L(a_i) = i$. Also notice that if x is on a face of Δ^n which is the convex hull of $\{a_{i_1}, \dots a_{i_k}\}$ then $L(x) \in \{i_1, \dots, i_k\}$. Thus the labelling of $\mathscr{T}_p$ satisfies the hypotheses of Sperner's Lemma. Thus, $\mathscr{T}_p$ contains a completely labelled n–simplex Δ_p. Choose $x_p \in \Delta_p$. Since Δ^n is sequentially compact, there is a convergent subsequence of $\{x_p\}$. To conserve notation, call this subsequence $\{x_p\}$ and let $L = \lim\{x_p\}$.

Claim: For each i, $L \in A_i$

By the choice of labelling Δ_p has a vertex w_p in A_i. Since $\operatorname{diam}(\Delta_p) \to 0$, $\lim w_p = \lim x_p = L$. A_i is the union of finitely many closed sets and so is closed. Thus, $L \in A_i$. □(Claim)

Since $L \in A_i$, there exists $U_i \in \mathscr{U}$ so that $\phi(U_i) = i$. Notice that $U_i \neq U_j$ if $i \neq j$. Thus, L is in the intersection of the sets $U_0, \dots, U_n$, and so $\mathscr{U}$ has order at least $n+1$, as desired. □

Theorem 4. The topological dimension of Δ^n is no more than n.

Proof. We must show that for all $\varepsilon > 0$, there exists a finite closed cover of Δ^n by sets of diameter less than epsilon which has order $n+1$. Such a cover can be constructed using barycentric subdivisions of Δ^n. We will not discuss the details here: look at page 59 in the text or see your class notes. □

Corollary 5. The topological dimension of Δ^n is exactly n.

Theorem 6 (Invariance of domain)**.** If an open set of $\mathbb{R}^m$ is homeomorphic to an open set of $\mathbb{R}^n$ then $m = n$.

Proof. Suppose that $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets and that $h\colon U \to V$ is a homeomorphism. Since U is open, it contains an open ball. Thus,

U contains an m–simplex Δ^m. Since h is continuous, $h(\Delta^m)$ is a compact set in $V \subset \mathbb{R}^n$. A set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Since $h(\Delta^m)$ is bounded, it is contained in a ball $B_r(0)$. Let $e_k = (0,\dots,0,r,0,\dots,0)$ with the kth coordinate equal to r. Let e_0 denote a vector not in $B_r(0)$ which is affinely independent from $\{e_1,\dots,e_n\}$. Then the convex hull of $\{e_0,\dots,e_n\}$ is a n–simplex Δ^n in $\mathbb{R}^n$ containing $h(\Delta^m)$. By an exercise, the topological dimension of Δ^n is at least the topological dimension of $h(\Delta^m)$. Since topological dimension is a homeomorphism invariant, this shows that $n \geq m$. The argument obtained by switching n and m in the previous argument shows that $m \geq n$. Thus, $m = n$. □