Physical Knot Theory

Based on

"A BRIEF INTRO TO KNOT THEORY FROM THE PHYSICAL POINT OF VIEW"
By Adams

and

"Physical Knot Theory: An Introduction"
by Millett
How do we model knots as physical objects?
- Thickness, Length
- Rigid segments / Lattice Knots

**Def**: Suppose \( \gamma \) is a simple closed curve in \( \mathbb{R}^3 \)

A tubular neighborhood is \( \{ x \in \mathbb{R}^3 \mid d(x, \gamma) \leq r \} \)

A thick knot is such a tubular neighborhood s.t.

\[ \{ x \in \mathbb{R}^2 \mid d(x, \gamma) = r \} \] is an embedded torus.

Its thickness is \( 2r \).

It's embedded if it doesn't intersect itself.

**Eg**: Tube crashes into itself.

For a knot type \( K \) (such as "trefoil")
the rope length (or just length) of \( K \) is

\[ l(K) = \inf \{ \ell \mid \exists \gamma \text{ a curve in } \mathbb{R}^3 \text{ of length } \ell \text{ having } N_{1/2}(\gamma) \text{ embedded} \} \]
Fact (Buck, Simon, Rawdon)

\[(\text{const}) \ C_r(CK)^{3/4} \geq \mathcal{F}(K) \geq 2.135 \ C_r(CK)\]

For no knot type is the rope length actually known

\[\mathcal{F}(\text{trefoil}) \approx 32.7/2\]
\[\mathcal{F}(\text{nontriv knot}) \approx 31.3/2\]
One strategy is to deform $K$ to lie on cubic lattice.

**Easy** to estimate length.

**Lemma (Diao & Ernst)**

If $B$ is a braid with $b$ strings and $n$ crossings, then its closure can be realized on cubic lattice with at most $12bn$ edges.

$\Rightarrow 4(b+1)n$ is length of open braid.

Need additional $4n+2$ edges to close for each strand.

Can assume $n > b-1$.

$\Rightarrow \text{length} \leq 8bn + 4n + 2b \leq 12bn$.

Gives $O(bn)$ as upper bound for $L$. 
Model physical knots as polygons in $\mathbb{R}^3$ e.g. atoms & bonds.

$\text{stick } # (\text{knot type}) = \min \# \text{ of edges in a polygon having that knot type}$

$\text{stick (unknot)} = 3$

$\text{stick (trefoil)} = 6$

A $(p, q)$ torus knot is a knot lying on a torus wrapping $p$ times one way and $q$ times the other.

It's known that if $p < q < 2p$ then

$\text{stick } # = 2q$

How?

Upper bounds we need a specific configuration
For a torus knot we can build one with two intersecting hyperboloids of different waist sizes.

These are ruled surfaces, so we can join points by straight lines lying in the surface (choosing the points carefully).

Doing this in both hyperboloids produces a stick configuration of a torus knot.

From Shapeways by mathgirll
If $\gamma$ is a simple closed curve in $\mathbb{R}^3$ and if $\vec{v} \in \mathbb{R}^3$ is a unit vector (i.e. $\vec{v} \in S^2$) then $b(\gamma, \vec{v}) = \# \text{ of maxima of } \gamma \text{ in the direction } \vec{v}$

Ex

- $b(\gamma, \vec{v}) = 2$

If $K$ is a knot type then $b(K) = \min \min b(\gamma, \vec{v})$ where $\gamma$ has knot type $K$.

Ex If $b(K) = 1$ then $K$ is the unknot

Proof $b(K) = 1 \Rightarrow K = O = \varnothing$

Ex If $b(K) = n$ the $K$ has a diagram of the form

- Let $\gamma, \vec{v}$ achieve the min
Def: Let $x$ be a simple closed curve in $\mathbb{R}^3$.
its super bridge # is $sb(x) = \max_{v \in S^2} (\# \text{maxima in } \langle v \rangle)$
assuming it is finite.
(i.e. not
not obvious
not obvious

for a knot type $K$

$sb(K) = \min_x sb(x)$

st. $x$ has knot type $K$.

Facts: $b(K) \leq sb(K) \leq 2b(K)$

$\uparrow$

obvious

$\uparrow$

not obvious

$\cdot$ $sb(K) \leq \text{stick #}(K)$
How to sample knot space?

Eg. How many distinct knot types represented by equilateral polygons of length \( n \)?

- HomFQPT used to distinguish
- bounded but unknown
- experimentally many knot types appear only once.

Conjecture (Cimmino)
On average, knotting & slipknotting is strongly local
if \( \text{Average length of knot} \leq \text{constant} \).