Coloring by groups

\[ Z/5Z \]

\[ 2a - b - c = 0 \]

\[ c = 2a - b \]

Reflections in dihedral group
Def A coloring of a diagram $D$ by a group $G$ is a function
\[ \{ \text{strands} \} \rightarrow G \]

At $a, b, c \in G$

$G = RBR$ by work on dihedral groups: Yes!
To be a coloring by $D_5$, we need:

$Z/5Z \downarrow$
reflections in $D_5$

$2 = 2 \cdot 4 - 1 \mod 5$
$= 8 - 1$
$= 7$
$= 2 \mod 5 \checkmark.$
Thm If a diagram $D'$ is obtained from a diagram $D$ by a Reidemeister non-trivial move if $D$ has a coloring by $G$ then so does $D'$.

$$b = a \cdot a^{-1} = a$$

$$\begin{cases} c = a \cdot b \cdot a^{-1} \Leftrightarrow b = a^{-1}c \cdot c \\ c = a \cdot d \cdot a^{-1} \Leftrightarrow d = a^{-1}c \cdot a \Leftrightarrow b = d \end{cases}$$
\[ c = aba^{-1} \]
\[ e = bdb^{-1} \]
\[ f = aeaa^{-1} \]

From the given equations:
\[ f = a b d b^{-1} a^{-1} \]
\[ b = a^{-1} ca \]

Using the Reidemeister moves:
\[ f = a^{-1} c a d a^{-1} c^{-1} a^{-1} \]
\[ = c a d a^{-1} c^{-1} \]

*Also check other Reidemeister versions*
Note the connection w/ $K$-colorings:

A diagram $D$ is $K$-colored $\iff D$ is colored by $D_K$ using reflections.

But using other groups we can get many more types of colorings!

Example: Use the group $\text{PSL}_2 \mathbb{C}$

$$\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$$

Bézier curves $(a, b, c, d) = (a, b, -c, -d)$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$\rightarrow$ Hyperbolic Geometry