Problems for Review on Taylor Series

For the Final, take another look at Problems 5 through 14
(all Taylor series and power series are centered at \( x = 0 \))

Find the interval of convergence of the series in Problems 1 through 3

1. \( \sum_{n=1}^{\infty} n^3 x^n \)
2. \( \sum_{n=1}^{\infty} \frac{x^n}{n^n} \)
3. \( \sum_{n=1}^{\infty} (-1)^n \frac{3^n x^n}{\sqrt{n}} \)

4. For an arbitrary positive integer \( q \), find the radius of convergence of \( \sum_{n=1}^{\infty} \frac{(n!)^q}{(qn)!} x^n \).

5. Find the Taylor polynomial \( T_3(x) \) of \( f(x) = \frac{1+x}{e^x} \)

6. Find the Taylor polynomial \( T_{10}(x) \) of \( \int \cos(x^2) \, dx \)

7. Find the Taylor polynomial \( T_5(x) \) of \( \sin(x) \cos(x) \)

Find the Taylor series of the functions in Problems 8 through 11

8. \( \frac{x}{1-x^2} \)
9. \( \int \frac{1}{1+x^2} \, dx \)
10. \( \frac{1}{2} (e^{x} + e^{-x}) \)
11. \( \ln(e + x) \)

12. For \( f(x) = \cos(x) \), find the Taylor polynomial \( T_4(x) \). Use your answer to estimate \( \cos(1) \), and find an error bound for your estimate.

13. Estimate \( \int_0^1 e^{-x^2} \, dx \) to within 0.01 (meaning that 0.01 is required to be an error bound).

14. For \( f(x) = \sqrt{1+x} \), find the Taylor polynomial \( T_3(x) \). Use your answer to estimate \( f(-0.5) = \sqrt{0.5} \), and find an error bound for your estimate.

15. Prove that \( 1 + x + \frac{1}{2} x^2 + \ldots + \frac{1}{n!} x^n + \ldots = e^x \) for negative \( x \) (in class we discussed the case of a positive \( x \)).

16. Find the Taylor polynomial of \( f(x) = e^{-3x} \) “from first principles”, computing all the \( f^{(n)}(x) \).

17. Find \( \sum_{n=0}^{\infty} (-1)^n \frac{2^{3n+2}}{3^{2n+3}} \)
Review on Series of Numbers:

Take another look at these series from Sections 1.3 and 1.4

1. \( \sum_{n=1}^{\infty} (0.7)^n \)  
2. \( \sum_{n=1}^{\infty} \frac{1}{n^{0.7}} \)  
3. \( \sum_{n=2}^{\infty} \frac{1}{(\ln(n))^n} \)  
4. \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}} \)  
5. \( \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^e} \)

6. \( \sum_{n=1}^{\infty} \frac{1}{n^{1/2} + n^{3/2}} \)  
7. \( \sum_{n=1}^{\infty} \left( \frac{3n+4}{5n+6} \right)^n \)  
8. \( \sum_{n=1}^{\infty} \frac{e^n n^3}{n!} \)  
9. \( \sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{n^2} \)  
10. \( \sum_{n=1}^{\infty} \frac{(n!)^2 + n!}{(2n)!} \)

11. \( \sum_{n=2}^{\infty} \frac{1}{n + (-1)^a \sqrt{n}} \)  
12. \( \sum_{n=1}^{\infty} \frac{n^a}{3^n (n!)} \)  
13. \( \sum_{n=1}^{\infty} (-1)^n \left( \sqrt{n+1} - \sqrt{n} \right) \)  
14. \( \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \)

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1 Hint: Use the conjugate.

2 Hint: \( \sqrt[n]{n} < 2 \) for all positive integers \( n \); you may use this result without proof.