Math 253, Spring 2001, Final Exam

Note: "Matrix A is diagonalizable" means: There is an eigenbasis for A.

1. True or False?
(In each part, you earn 2 points for the correct answer, and 1 point if you don't answer.)

T F The algebraic multiplicity of an eigenvalue cannot exceed its geometric multiplicity

T F If 4 is an eigenvalue of a 4×4 matrix A, then 4^4 must be an eigenvalue of the matrix A^4.

T F If a matrix A is diagonalizable, then A must be invertible.

T F If a 10×10 matrix has 5 distinct eigenvalues, then the rank of A must be at least 5.

T F All triangular matrices are diagonalizable.

2. True or False?
(In each part, you earn 2 points for the correct answer, and 1 point if you don't answer.)

T F The equation $\det(A)=\det(-A)$ holds for all square matrices.

T F The function $T\left(f(x)\right)=f''(x)-3f'(x)+2f(x)$ is a linear transformation from C^∞ to C^∞.

T F If A is an invertible 4×4 matrix, then the equation $\operatorname{rank}(A)=\operatorname{rank}\left(A^{-1}\right)$ must hold.

T F Let V be the set of all 2×2 matrices A such that the vector $\begin{bmatrix}1\\2\end{bmatrix}$ is in the image of A. Then V is a subspace of $\mathbb{R}^{2\times 2}$.

T F If V is a one-dimensional subspace of $\mathbb{R}^3$, then there is a 2×3 matrix A such that $V=\ker(A)$.

3. Find the matrix B of the linear transformation $T(\vec{x})=\begin{bmatrix}-1 & 2\\ 4 & 1\end{bmatrix}\vec{x}$ with respect to the basis $\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}$.

basis $\begin{bmatrix} -1 \end{bmatrix}, \begin{bmatrix} 2 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 3 & 9 \\ 1 & 0 & 2 & 0 & -1 \end{bmatrix}$

a. Find a basis of the image of matrix A.
b. Find a basis of the kernel of matrix A.

5.a. For which values of the constants a and b does the matrix $\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$ have the eigenvalues 2 and 3?

b. Find nonzero values of a and b such that the matrix $\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$ fails to be diagonalizable. (You are asked to find only one value for a and one value for b.)

6. Consider the linear transformation $T(f(x)) = f(x-2)$ from P_2 to P_2.
a. Find the matrix of T with respect to the standard basis $1, x, x^2$ of P_2.
b. Find all the eigenfunctions of transformation T, that is, the functions $f(x)$ such that $T(f(x)) = \lambda f(x)$ for some scalar λ.

7. For which values of constants b and c is the matrix $A = \begin{bmatrix} 1 & 1 & b \\ 0 & 3 & c \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable?

(Find all solutions, expressing c in terms of b).

8a. Three holy men who put little stock in material things (let's call them Abdullah, Benjamin, and Chaim), get together each day for the following bizarre ritual: Each of them takes his whole fortune out of his bundle and gives it away to the two others, in equal parts. For example, if Abdullah has 6 Francs one day, he will give 3 Francs each to Benjamin and Chaim. If Abdullah starts out with 6 Francs, Benjamin with 1 Franc, and Chaim with 2 Francs, find formulas for the amounts $a(t), b(t)$, and $c(t)$ each will have after t distributions. (Between rituals, none of them ever earns or spends any money).

after 7 distributions. (Between rituals, none of them ever earns or spends any money).

Hint: The matrix $\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}$ has the eigenvectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}$, and $\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

b. Who will have the most money after one year, that is, after 365 distributions?

9. Let V be the space of all 2×2 matrices A such that $A\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}A$

a. Find a basis of V, and thus determine $\dim(V)$.

b. Find the dimension of the image of the linear transformation
$T(A)=A\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}-\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}A$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

10. Consider the linear transformation $T\left(f(x)\right)=f'(x)\cdot x$ from P to P (the space of all polynomials).

a. Find the kernel of T and its dimension.

b. Find a polynomial that fails to be in the image of T.

c. Find all the eigenfunctions and eigenvalues of T.