Math 253, Fall 2001, Final Exam

1. True or False?

a. If vector $\vec{b}$ is in the image of matrix A, then the linear system $A\vec{x}=\vec{b}$ must be consistent.

b. If A is a 4×3 matrix, and $\vec{v}_1,\vec{v}_2,\vec{v}_3$ are linearly dependent vectors in $\mathbb{R}^3$, then the vectors $A\vec{v}_1,A\vec{v}_2,A\vec{v}_3$ must be dependent as well.

c. Matrix $\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$ is similar to matrix $\begin{bmatrix}4 & 1\\ 3 & 2\end{bmatrix}$.

d. If the product of two square matrices A and B is invertible, then A must be invertible as well.

e. If the determinant of a square matrix A is 1 or -1, then A must be an orthogonal matrix.

2. True or False?

a. If a 10×10 matrix A has 6 distinct eigenvalues, then the rank of A must be at least 5.

b. If matrix A is invertible, then there must be an eigenbasis for A.

c. If the characteristic polynomial of a 7×7 matrix A is $f_A(\lambda)=\lambda\left(\lambda^2-1\right)\left(\lambda^2-2\right)\left(\lambda^2-3\right)$, then there must be an eigenbasis for A.

d. If 1 is the only eigenvalue of a diagonalizable $n\times n$ matrix A, then A must be the identity matrix I_n.

e. If $\vec{v}$ is an eigenvector of A^2, then $\vec{v}$ must be an eigenvector of A as well.

3. Find the matrix B of the linear transformation $T(\vec{x})=\begin{bmatrix}3 & 2\\ 4 & 5\end{bmatrix}\vec{x}$ with respect to the basis $\begin{bmatrix}1\\ 2\end{bmatrix},\begin{bmatrix}-1\\ 1\end{bmatrix}$.

4. a. Find the determinant of the matrix $A=\begin{bmatrix}1 & -1 & 7 & 0\\ 0 & 0 & 6 & 0\\ 3 & 0 & 9 & 4\\ 0 & 5 & 10 & 6\end{bmatrix}$

$$\begin{bmatrix}0 & 5 & 10 & 6\end{bmatrix}$$

b. Is matrix A invertible?

5. Let $$A=\begin{bmatrix}1 & 2 & 0 & 0 & 3\\0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 4\\1 & 2 & 0 & -1 & 6\end{bmatrix}$$

a. Find a basis of the image of matrix A.
b. Find a basis of the kernel of matrix A.

6. A rabbit population and a wolf population are modeled by the equations

$$\begin{aligned}p(t+1) &= 6p(t) - 2q(t)\\ q(t+1) &= p(t) + 3q(t)\end{aligned}$$

The initial populations are $p(0)=600$ and $q(0)=500$.

a) Which are the rabbits and which are the wolfs?
b) Find closed formulas for $p(t)$ and $q(t)$.

7. Consider the linear transformation $L(A)=2A+3A^T$ from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$.

a. Find the matrix B of transformation L with respect to the standard basis of $\mathbb{R}^{2\times2}$.
b. Is transformation L an isomorphism?
c. Is the identity matrix I_2 in the image of L?

8. Let V be the space of all 2×2 matrices A such that the vector $\begin{bmatrix}1\\2\end{bmatrix}$ is in the kernel of A.

a. Find a basis of V, and thus determine $\dim(V)$.

b. Find the dimension of the space W of all 2×2 matrices A such that $\begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector of A. (Hint: Using part a, you can answer this question without much computational work.)

9. a. Is there a 2×2 matrix A with eigenvalues 4 and 6, such that all the four entries of A are positive? Give an example of such a matrix A, or explain why none exists.
b. Is there a 2×2 matrix B that fails to be diagonalizable, such that all the four entries of B are positive? Give an example of such a matrix B, or explain why none exists.

10. Let P be the space of all polynomials, and let V be the space of all infinite sequences of real numbers. Consider the linear transformation

$$T(f(x))=(f(0),f(1),f(2),\ldots,f(n),\ldots) \text{ from } P \text{ to } V.$$

a. Find the kernel of T. Explain your answer carefully.

a. **Find the kernel of T. Explain your answer carefully.**

b. **Is T an isomorphism? Explain your answer carefully.**