MA434—Topics in Abstract Algebra

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Abstract algebra has become the language of much of mathematics. Everywhere you look in more advanced mathematics, you see both the language (groups, rings, vector spaces, homomorphisms, isomorphisms, ...) and the spirit (abstraction, generality, reducing things to their essential axiomatic description) of algebra. You have all taken a first course in algebra, and have begun to learn to speak this language. The point of this course is to move you just a little bit more down that road.

This "topics" course allows us to change the exact subject on which to focus, and it can be retaken for credit. This year, we will be focusing on Algebraic Geometry, which is the study of the geometric objects defined by polynomial equations. You have been familiar with some of these for a long time: $x^2 + y^2 = a$ defines a circle in $\mathbb{R}^2$, and $ax+by = c$ is a line. These are some of the simplest examples of algebraic curves, and it is easy to create more by using polynomials of higher degree. For more curves we could also move to $\mathbb{R}^3$ and require two different equations to hold; for example, $y = x^2$, $z = x^3$ defines a curve. Or we could write one equation in three variables to define a surface; you might remember the gallery of quadric surfaces from multivariable calculus. In general we study *algebraic varieties*, which are the solution sets of systems of polynomial equations.

One interesting feature of algebraic varieties is that they can be *singular*, i.e., they do not always look like smooth curves. The equation $x^2 - y^2 = 0$, for example, gives two crossing lines. If you plot $y^2 = x^3$, you will see it has a cusp. Studying algebraic varieties pretty much forces us to study singularities as well.

Since our objects are defined by polynomial equations, they make sense in a far more general setting. We could, for example, work over a field different from $\mathbb{R}$: what can we say about the *complex* solutions of $x^2 + y^2 = a$, or about the solutions in $\mathbb{Z}/p\mathbb{Z}$? It even makes sense to ask for solutions where x and y are supposed to be polynomials in $\mathbb{Q}[t]$, which would amount to finding a *family* of solutions parametrized by t.

In order for the theory to work over arbitrary fields (or rings), we need to avoid calculus. We know, for example, how to find a tangent line to the circle using derivatives. Does that still make sense over $\mathbb{Z}/p\mathbb{Z}$? In a situation with no pictures, what does "tangent" even mean? A lot of the challenge of algebraic geometry comes from the need to study geometric objects using only algebraic techniques.

Algebraic geometry has a very long history. It goes at least as far back as Descartes, who first made the link between equations and curves and who also argued that *only* algebraic curves were worthy of mathematical study. Newton wrote a paper classifying all possible plane curves of degree 3. But the subject really took off in the 19th century. One of the crucial discoveries of that century was that the theory became considerably simpler when one worked in projective space and with an algebraically closed field. In the late 19th century algebraic geometry really meant complex projective geometry.

The 20th century saw another huge leap when people realized that there were other fields besides $\mathbb{C}$ to consider. This was largely motivated by number theory: many classical problems can be reinterpreted in terms of algebraic varieties over $\mathbb{Z}$ or over finite fields. Recasting the theory in a way that allowed for that took a lot of work that eventually culminated in a whole new way of doing algebraic geometry discovered by Grothendieck in the 1960s. Algebraic varieties were replaced by much more general objects called *schemes*.

Modern algebraic geometry is difficult (maybe even very difficult). It relies on so much preliminary work in commutative algebra and sheaf theory (at least) that if we tried to go that route we would never get past the preliminaries. In this course, then, we will follow Miles Reid and proceed another way, focusing on examples that are simple enough to be handled with more elementary means. The idea is to build up your intuition and your taste. Perhaps that will help motivate you to learn more.

In addition to understanding fields, the main pre-requisite from abstract algebra is ring theory. Most of the time we will be working with rings of polynomials (in several variables) with coefficients in a field. Quotients of those rings will also show up.

Reid's book was first published in 1988, so the one thing it ignores completely is mathematical software. There was little that was useful then, but today programs such as *Sage* make it easy to do computations. (There is a whole field of "computational algebraic geometry" dedicated to thinking about doing such computations in practical ways.) Computations don't just help us solve specific problems; they also bring insight. A good example often clarifies what is going on.

I'm hoping you will enjoy the course and the material. Feel free to talk to me if you think things are going badly or if you have any concerns.

Goals of this Course: What do I expect you to be able to do after this course? Of course, I want you to learn some elementary algebraic geometry. But here are some other goals:

- I want to give you the experience of getting deeply engaged with a mathematical topic. (I hope it'll be an enjoyable experience.)
- I hope you will further develop your ability to understand, critique, and create mathematical arguments.
- I hope you will leave the course more confident in your ability to think creatively and to solve problems.

- I hope you will improve your ability to think precisely.
- I hope you will improve your ability to *speak* and *write* about mathematics.
- I hope you will learn to use *Sage* both to do computations and to provide insight.
- I hope you will understand the basics of Algebraic Geometry in a way that prepares you to go further.
- I hope that by putting it to use you will end up with a better grasp of the whole span of mathematics you have learned so far.
- I hope you will have better control of the abstract algebra you learned in MA333; in particular, I hope you will be ready to use it in other mathematical contexts.

Where to find me: Here's the basic information:

office: Davis 208
phone: 5836
email: fqgouvea

If you need to reach me when I'm not in my office, email is the best method. I'm usually pretty quick at answering email.

Course web site: I've been too lazy to learn to use Moodle, so I'll just post stuff (including assignments) here:

`http://www.colby.edu/~fqgouvea/434`

Please visit and bookmark it.

How the Class Will Work: After a couple of introductory classes, this course will be *entirely based on student presentations supported by student writing*. For each class, I will assign a topic for one of you to present, and I will also choose one of you to be the "scribe" for the day.

Presentations will be assigned in pairs. I will post on the course web site a portion of the textbook to read, understand, and explain to the class. Each pair can then decide how to divide the work. Each student should get to present at least part of the material. You should expect questions and interaction from the rest of the class.

The job of the scribe is to take notes and to write them up for the rest of the class. I will post these on the course web site as well, so you should turn in your scribal report in electronic form. Scribal reports are due **on or before the next class session**.

Because each bit of mathematics builds on the previous bit, it is extremely important to keep up with what is being presented. So even if it is not your turn to speak, you should be doing the reading and attempting to solve problems. This will allow you to ask more intelligent questions as well.

The key element of an advanced course like this one is your *engagement* with the mathematics. For it to work well for you, you need to put in the effort.

Office hours: I will be in my office and available for questions, discussion, and general conversation on **Mondays and Wednesdays** from 1:30 to 3:30 PM. You may arrive at my office to find me busy with another student. Please be patient; I'll try to make sure that everyone gets a chance to see me.

You may find me in my office at other times, and if I have the time I will be happy to talk to you. But only at the times above do I promise to be available. If you really need to see me at another time (except on Thursdays, which are verboten), send me an email and we'll set up an appointment.

Please do not hesitate to come see me—in fact, I strongly encourage you to come. It is part of your education, and one of your privileges as a Colby student.

Problem Sets: To learn mathematics, we need to solve problems. So there will also be problem sets, but they will be small. Rather than turning in written solutions, however, we will discuss them in class. The scribe for the day will have the task of writing up these solutions, so you can help them by having written solutions you can share.

Exams: There will be two take-home exams. The midterm exam will be on the week of March 9–13. The final exam will be during exam period.

Grading: Your grade will be computed as follows:

presentations	25%
scribing	25%
problems	10%
midterm	20%
final	20%

Notice that more than half of your grade will come from what happens in class: your presentations and problem solving.

Attendance: Given that almost all the action will take place in class, I hardly need to tell you that you are expected to be there. Should you be unable to come to class on a particular day, please contact me. Unexcused absences *will* have an effect on your grade for the course. Too many unexcused absences will lead to your being asked to withdraw from the course.

Textbooks: We will use *Undergraduate Algebraic Geometry*, by Miles Reid, as our basic textbook. It seems small, but as you will see it will become much larger when we start unpacking it.

Other books: Keep your abstract algebra textbook handy. For some of the more advanced background in algebra, without proofs, it might be worthwhile to look at my *Guide to Groups, Rings, and Fields*.

There are many more algebraic geometry books, almost all of which are too difficult. Here are some worth looking at:

- *Plane Algebraic Curves*, by Egbert Brieskorn and Horst Knörrer, is a big book full of classical results and pictures. Not easy reading, but fun.
- The first chapter of *Elementary Algebraic Geometry*, by Keith Kendig, is very much worth reading for its elementary description of complex projective curves.
- *Algebraic Geometry: A Problem Solving Approach*, by a crowd of people led by Tom Garrity, is great for self-study.
- *Introduction to Algebraic Geometry*, by Brendan Hassett, is an introductory book at the undergraduate level. It foregrounds the computational side of commutative algebra.
- *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, by David Cox, John Little, and Donal O'Shea, also starts with computational commutative algebra.
- Amon Neeman's *Algebraic and Analytic Geometry* is an attempt to teach scheme theory at an undergraduate level.
- My first course was with Joe Harris, who later turned it into the textbook *Algebraic Geometry*. It is very geometric, but not easy.
- The first volume of I. R. Shafarevich's *Basic Algebraic Geometry* is a graduate introduction to the kind of algebraic geometry we will be doing.
- Robin Hartshorne's *Algebraic Geometry* is the standard graduate textbook on full-blown modern algebraic geometry. It is very difficult.

The point of this long list should be clear: one of the best ways to understand mathematical ideas is to look at different proofs, different points of view, different styles. Read widely!

Software: There are two very useful (and free) programs you should learn to use: *Sage* and GP. On the course web site there is a handout with a (very) short introduction to these programs. For our purposes *Sage* is probably more useful, though I tend to use GP for quick computations.

Academic Honesty & Consequences for Academic Dishonesty: You are encouraged to interact with others as you tackle your assigned section and work on the problem sets (in fact, it will be a lot more fun doing it that way). And of course you are expected to interact in class as well. The paper, however, must be your own. You may, of course, seek help from any and all sources, but in the end what you write must be a result of your own thought processes and your grappling with the material.

Honesty, integrity, and personal responsibility are cornerstones of a Colby education and provide the foundation for scholarly inquiry, intellectual discourse, and an open and welcoming campus community. These values are articulated in the *Colby Affirmation*. I will expect you to behave accordingly. You are expected to demonstrate academic honesty in all aspects of this course. If you are clear about course expectations, give credit to those whose work you rely on, and submit your best work, you are highly unlikely to commit an act of academic dishonesty.

Academic dishonesty includes, but is not limited to: violating clearly stated rules for taking an exam or completing homework; plagiarism (including material from sources without a citation and quotation marks around any borrowed words); claiming another's work or a modification of another's work as one's own; buying or attempting to buy papers or projects for a course; fabricating information or citations; knowingly assisting others in acts of academic dishonesty; misrepresentations to faculty within the context of a course; and submitting the same work, including an essay that you wrote, in more than one course without the permission of the instructors.

Academic dishonesty is a serious offense against the college. Sanctions for academic dishonesty are assigned by an academic review board and may include failure on the assignment, failure in the course, or suspension or expulsion from the College. For more on recognizing and avoiding plagiarism, see the library guide: `libguides.colby.edu/avoidingplagiarism`.

Finally: Have fun. We're going to be studying some neat stuff, and we'll be doing it in a way that's interactive and dynamic. I'm planning to enjoy it, and I hope you do too.

> If you grant a small favour to a lover, you will soon afterward have to grant him more, and eventually that can go far. In the same way, grant a mathematician the slightest principle and he will draw a consequence from it that you'll have to grant him too, and from that consequence yet another; and in spite of yourself, he will lead you farther than you would ever have thought possible.
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> – Bernard Le Bovier Fontenelle