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MA434 Class Notes for 24 February 2020: Linear Systems

§ 2.4

We began class with section 2.4, discussing the vector space of forms (homogeneous polynomials) and looking for its dimension.

Joshua defined S_d as the set of forms $\{F : F \text{ is a form of degree } d \text{ in } (X, Y, Z)\}$. Given some form $F \in S_d$, we can write $F = \sum a_{ijk} X^i Y^i Z^k$, where the sum is taken over i, j, k with $i + j + k = d$, with all $i, j, k \geq 0$, and with coefficients a_{ijk} in some field k.

Now we call S_d a k-vector space with the field k, and it has a basis $v_1 \cdot v_2 \cdots v_d$ where $v_i = X$ or Y or Z. To find the dimension of the vector space, we take $v_1 \cdot v_2 \cdots v_d$ and multiply it by XYZ. We see that $v_1 \cdot v_2 \cdots v_d XYZ$ now has $d + 3$ variables, some X, some Y, and some Z. Essentially trying to find the number of monomials of this degree, which is why we multiplied by XYZ. Taking $v_1 \cdot v_2 \cdots v_d XYZ$, Joshua suggests we consider the basis à la stars and bars:

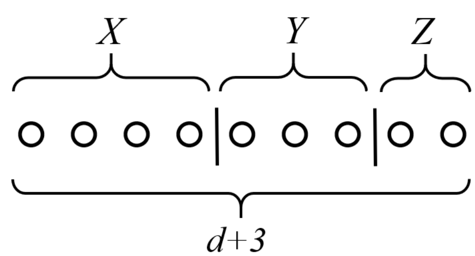

Figure 1. Partitions of $d + 3$ into three parts.

In this format, with $d + 3$ being multiplied by the choice of either X, Y or Z, it's easy for us to see that the $d + 3 - 1 = d + 2$ bases can be partitioned in two places, meaning that there are $\binom{d+2}{1}$ possibilities. Reid alternatively offers a triangular demonstration of how the same number of bases can be arrived at. However we go about it, we find the dimension of the vector space dim $S_d = \binom{d+2}{2}$.

Now, taking $S_d(P_1, P_2, ..., P_n) = \{F \in S_d : F(P_i) = 0 \text{ for } i = 1, 2, ..., n\}$, which Reid notes is a subset of S_d, Joshua showed that dim $S_d(P_1, P_2, ..., P_n) \geq$ dim $S_d - n = \binom{d+2}{2} - n$.

§ 2.5

Next we moved on to section 2.5, where Ethan laid out our first lemma: given some infinite field k and forms $F \in S_d$, if there is a line L in the projective space $\mathbb{P}^2_k$ such that $F \equiv 0$ on L, then given the equation H of the line L, $F = H \cdot F'_{d-1}$; this can be visually represented by a proverbial transformation of the X-axis to the line L, as seen in Figure 2.

For the proof, we will first call $H = X'$, and $F(X', Y', Z')$ We then consider the equation $F = X'(??) + G(Y', Z')$, and look to see what $H = X'$ is multiplied by in the equation of F. Ethan pointed out that the unknown factor with X' must be of degree $d - 1$, and so we proceeded to write the equation as $F = X'F'_{d-1} + G(Y', Z')$. Ethan showed us that knowing $F = X'F'_{d-1} = 0$ for $0, Y', Z'$ on L, we see that as G has degree d and as L has infinitely many points, then $G(Y', Z') = 0$

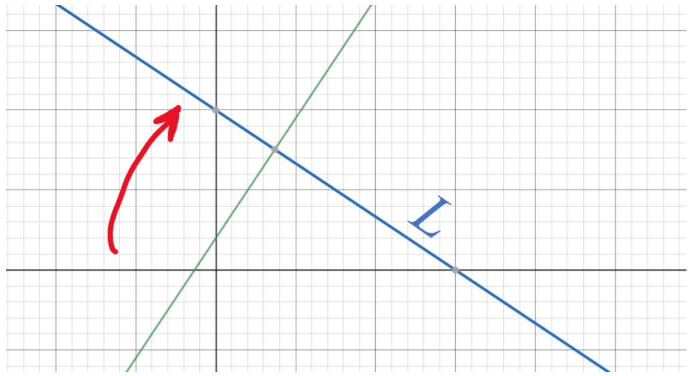

FIGURE 2. Visualizing the graphically representative transformation to a new X-axis on the line L with equation H.

and $F = HF'_{d-1}$. This is to say that the line H is contained inside this degree d curve and that the line divides the curve; the form "vanished" over the line.

Moving on from the form "vanishing" over the line to the form "vanishing" over a conic, we continue on to the second part of lemma 2.5. Again, we have the infinite field k and the form $F \in S_d$, with the non-degenerate conic $C \subset \mathbb{P}^2_k$. The conic C defined by the equation $C : Q = 0$.

F must be of the form $F = Q(??) + Y(??) + (??)$, and ideally we want it of the form $F = QF'_{d-2}$, so we'll write it as $F = QF'_{d-2} + YA(X, Z) + B(X, Z)$. We know that we can change coordinates and write $Q = XZ - Y^2$ and $Y^2 = XZ - Q$. We then parameterize the conic $C : [U^2 : UV : V^2]$ with $X = U^2, Y = UV, Z = V^2$. When $F \equiv 0$, we substitute, and we know by definition that $UVA(U^2, V^2) + B(U^2, V^2) = 0$, meaning $UVA(U^2, V^2) = -B(U^2, V^2)$, and because the powers of U and V don't match up, this must be the zero polynomial, making $U, V = 0$ and that the form F factors and that F can be written $F = QF'_{d-2}$.

Note that this proof only worked because we have an explicit parameterization of the conic, which we learned from section 1.7.

Ethan next presented a corollary to this lemma:

L defined as the line $L : (H = 0)$, L a line in $\mathbb{P}^2_k$. Given points $P_1, ..., P_a \in L$ and $P_{a+1}, ..., P_n \notin L$, with $a > d$, then $S_d(P_1, ..., P_a) = H \cdot D_{d-1}(P_{a+1}, ..., P_n)$.

We already proved that if the number of intersections is less than or equal to $2d$. Knowing that F is a form of degree D and that $a > d$ implies that $\forall [X : Y : Z]$ where $H[X : Y : Z] = 0$, then $F[X : Y : Z] = 0$. We know by the first lemma that we can factor out H from the form, so $F = H \cdot F'_{d-1}$. We know $H = 0$ on the first part, for $F'_{d-1} = 0$ on $\{d_{a+1}, ..., d_n\}$, so $F'_{d-1} \in S_{d-1}(P_{a+1}, ..., P_n)$.

The second part of the corollary takes the conic C defined by the quadratic $C : Q \equiv 0$, and we let $a > 2d$. So we know $S_d(P_1, ..., P_a) = Q \cdot S_{d-2}(P_{a+1}, ..., P_n)$, and the proof follows identically as the previous one with the line and was not produced in class (it is not in the book either).

For example, take the following line in figure 3: four of the points are on the line, so we could say $Q = H \cdot S_1(P_5)$. Essentially, we strive to show that the maximum number of points a conic can share with a curve of degree d is $2d$.

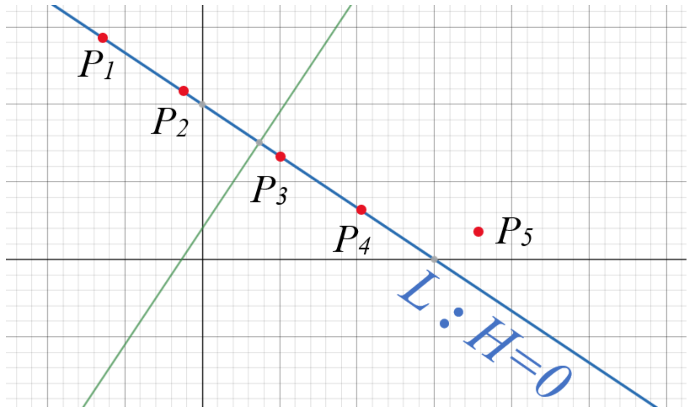

FIGURE 3. Showing the first four points $P_1, ..., P_4$ on a line $L : H = 0$, but the fifth point P_5 not on L because $5 > 2 \cdot 2$.

§ 2.6

Theorem: Let k be an infinite field and let the distinct points $P_1, ..., P_8 \in \mathbb{P}^2_k$. If no four points are collinear, and no seven points are conconic (lying in the same nondegenerate conic), then dim $S_3(P_1, ..., P8) = 2$. We'll prove in this section that dim $S_3(P_1, ..., P8) \leq 2$, and in 2.7 we will prove that the dimension is greater than or equal to two, thus showing equality.

There are three cases that we need to prove for this inequality:

Case 1: No three points are collinear, no six points are conconic.

We suppose, for the sake of contradiction, that dim$S_3(P_1, ..., P8) \geq 3$ and let P_9, P_{10} be distinct points on the line between P_1 and P_2, $\overline{P_1P_2}$. Then we see that the dimension dim $S_3(P_1, ..., P_{10}) \geq$ dim $S_3(P_1, ..., P_8) - 2 \geq 1$. Then there exists some form $F \neq 0$ where $F \in S_3(P_1, ..., P_{10})$, which is true because the dimension is at least one.

By section 2.5, we know there are four points on this line, and because four is more than the degree of the forms (being three), we again know that we can write the form F as the product $F = H \cdot Q$, where $Q \in S_2(P_3, ..., P_8)$.

If Q is nondegenerate, then the points $P_3, ..., P_8$ are conconic, and thus contradict our assumption of no six points being conconic. $\lightning$

If Q is degenerate, then three points of $P_3, ..., P_8$ must be collinear, thus contradicting our assumptions of no three points collinear. $\lightning$

Thus, for both nondegenerate and degenerate conics, we have contradictions when dim $S_3(P_1, ..., P8) \geq 3$, so we conclude that for this case, dim $S_3(P_1, ..., P8) \leq 2$.

Case 2: Three points are collinear.

Let points P_1, P_2, P_3 be in L, and take a point P_9 on L, distinct from the previous eight points. By section 2.5, we know that $S_3(P_1, ..., P_9) = H \cdot S_2(P_4, ..., P_8)$, and since no four points are collinear, then dim $S_2(P_4, ..., P_8) = 1$; we base this conclusion in part off of our conclusions from section 1.11, as there are five points, which can exist in a one dimensional space. We see that dim $S_3(P_1, ..., P9) = 1$, we remove the point P_9 and get at most one dimension back, so then dim $S_3(P_1, ..., P8) \leq 2$.

Case 3: Six points are conconic.

We begin by letting the six points $(P_1, ..., P_6)$ be on the conic C. We take the point P_9 on C, distinct from the previous eight points. We know the degree of the conic C is 3, and so $7 > 2d = 6$; then from our conclusion in section 2.5, we see that $S_3(P_1, ..., P_9) = QS_1(P_7, P_8)$. As we all know that the number of unique lines through two points is one, Joshua pointed out that $S_3(P_1, ..., P_9)$ has dimension 1, and again we remove the point P_9 and gain at most one dimension, to find that dim $S_3(P_1, ..., P8) \leq 2$.

Thus we can conclude that dim $S_3(P_1, ..., P8) \leq 2$.

At the beginning of next class, we will walk through section 2.7 to show the other inequality and conclude equality.