Math 434: Class Notes February 21, 2019
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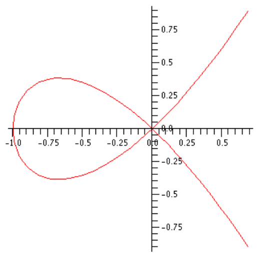
Section 2.1[1]

Nodal Cubic

A cubic represented by $C : (y^2 = x^3 + x^2) \in \mathbb{R}^2$.



It can be parameterized in the following way:

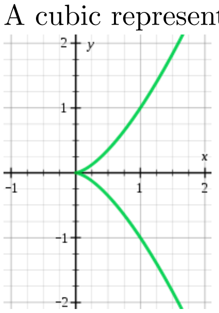
$$\rho : \mathbb{R}^1 \to \mathbb{R}^2$$

$$t \to (t^2 - 1, t^3 - t)$$

Looking at the graph, we can see that this parameterization makes sense. The x-values follow a quadratic sort of shape with an intersection at -1, hence $x = t^2 - 1$. Plugging this equation for x into the original equation, we get $y = t^3 - t$ as the parameterization of y.

Cuspidal Cubic

A cubic represented by $C : (y^2 = x^3) \in \mathbb{R}^2$.



Here, we write the parameterization:

$$t \to (t^2, t^3)$$

Again, looking at he graph we see that this parameterization makes sense.
A note from Fernando: We can derive these parameterizations starting with the fact that a line crosses a cubic three times. In the nodal cubic, it crosses at 0 twice, and we can vary t as the slope to get the other point. In the cuspidal cubic, you have to worry about how many times it intersects the cusp.

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[1]Graphs of the cubics in this section found on math.purdue.edu. They are the same as what was put on the board in class, but are nicer to look at than my drawings.

Section 2.2

Claim : The curve $C : y^2 = x(x-1)(x-\lambda)$ has no rational paramterization.
Theorem : Let k be a field of characteristic not equal to 2 and $\lambda \in k$ with $\lambda \neq 0$ or 1. Let $f, g \in k(t)$ be rational functions such that

$$f^2 = g(g-1)(g-\lambda).$$

Then, $f, g \in k$ (Note: $k(t)$ is the field of fractions of the UFD $K[t]$).

Proof. We have $f, g \in k(t)$ so we can write them as fractions. Let $f = \frac{r}{s}$ and $g = \frac{p}{q}$ where $r, s, p, q \in K[t]$ (polynomials in t) with r, s are coprime and p, q are coprime. We can rewrite our equation and manipulate it:

$$(\frac{r}{s})^2 = \frac{p}{q}(\frac{p}{q} - 1)(\frac{p}{q} - \lambda)$$

$$r^2 = \frac{s^2 p}{q}(\frac{p}{1} - 1)(\frac{p}{q} - \lambda)$$

$$q^3 r^2 = s^2 p(p-q)(p-\lambda q)$$

Since r, s are coprime, we must have that s^2 divides q^2. Similarly, since p, q are coprime, q^3 must divide s^2. Thus we have $s^2 = aq$ where a is a unit in $K[t]$ (note: the units in $K[t]$ are the nonzero elements of k). We now write

$$\frac{s^2}{q^2} = aq$$

$$(\frac{s}{q})^2 = aq$$

So, aq is a square in $K[t]$. We can now rewrite our previous equation, plugging in $s^2 = aq^3$ to get:

$$q^3 r^2 = aq^3 p(p-q)(p-\lambda q)$$

The q^3's cancel, giving us

$$r^2 = ap(p-q)(p-\lambda q)$$

Because p, q are coprime, we have that $ap, p-q, p-\lambda q$ are all relatively prime. Since their product is a square, each term is itself a square. Because p, q must be constants, as will be shown in the next section, it follows that r, s must also be constants, and we are done. □

Lemma 2.3

Let k be an algebraically closed field, p, q coprime elements in $K[t]$ and assume there exists λ_i, μ_i such that $\lambda_i p + \mu_i q$ is a square in $K[t]$ for $i = 1, 2, 3, 4$. Then $p, q \in k$ (for $\lambda_i : \mu_i \in \mathbb{P}^1$).

Proof. First we will show that if we have four linear combinations that are all squares, they can be written in the form $p, p-q, p-nq, q$. Considering hitting any linear combination with an invertible $2x2$ matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

where $a, b, c, d \in k$. So, four linear combinations that get hit with this matrix sill be linear combinations and maximal degree does not change. Assume we have the following:

$$ap + bq = p'$$

$$cp + dq = q'$$

and two more linear combinations. Then $p', q', \alpha p' + \beta q', \gamma p' + \delta q'$ are all squares. We can then multiply each of these by some of $\alpha, \beta, \gamma, \delta$ to get what we want:

$$\alpha\gamma p' = p$$

$$-\beta\gamma g' = q$$

$$\alpha\gamma p' + \beta\gamma q' = p - q$$

$$\alpha\gamma p' + \alpha\delta q' = p - \lambda q$$

Now that we have shown that four linear combinations can be expressed in this form, we can proceed to prove by our lemma by contradiction. Assume by contradiction that $deg(p) \neq 0$, $\deg(q) \neq 0$ and that p, q are minimal. We know that p, q are squares and we can write $p = u^2$ and $q = v^2$. We also have

$$p - q = (u + v)(u - v)$$

$$p - nq = u^2 - nv^2 = (u - \sqrt{n}v)(u + \sqrt{n}v)$$

(note: $deg(u) < deg(p)$ and $deg(v) < deg(q)$ because we assume nonzero degree by contradiction, and u, v are coprime because p, q are coprime). Now, we will show that $u - v, u + v$ are coprime and $u - \sqrt{n}v, u + \sqrt{n}v$ are coprime.

Assume that $d|(u + v)$ and $d|(u - v)$. Then $d|(u + v + u - v) = 2u$ and $d|(u + v) - (u - v) = 2v$. Since 2 is a unit, this means that d divides both u and v, but we have assumed that d, u are coprime, so $d = 1$. We can use the same argument to show that $u - \sqrt{n}v, u + \sqrt{n}v$ are coprime. Thus, we have that $u + v, u - v$ are relatively prime and their product is a square, and the same is the case for $u - \sqrt{n}v, u + \sqrt{n}v$. However, this is a contradiction because u and v have smaller degree than p and q, which we assumed to be minimal. So, $p, q \in k$.

(note: pay attention to how we used the crucial facts that we are working in an algebraically closed field and that k did not have characteristic equal to 2) □

Summary:

The main idea that we covered today is that the nodal cubic (2 distinct roots) and the cuspidal cubic (1 distinct root) can be parameterized, while $C : y^2 = x(x - 1)(x - \lambda)$, which has 3 distinct routes, can not be parameterized.