**♡ Topics in Abstract Algebra: Valentines Day ♡**

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Annie and Lanie's Part

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**Last Time (1.9):**
If L is a line in $\mathbb{P}^2$ and D is a curve of degree d, then L and D intersect at most d times. Similarly, if C is a nondegenerate conic in $\mathbb{P}^2$, then C and D intersect at most 2d times.

**Corollary (1.10):**
Let $\heartsuit_1, \heartsuit_2, \heartsuit_3, \heartsuit_4, \heartsuit_5 \in \mathbb{P}^2$ be distinct points such that no four points are collinear. There exists at most one conic that goes through all $\heartsuit_1, \heartsuit_2, \heartsuit_3, \heartsuit_4, \heartsuit_5$.

**Proof:**
For the sake of contradiction, let $C_1 \neq C_2$ be conics that contain all five points. Thus, $\{\heartsuit_1, \heartsuit_2, \heartsuit_3, \heartsuit_4, \heartsuit_5\} \subset C_1 \cap C_2$.

*Case 1:* Both $C_1$ and $C_2$ are nondegenerate.
Since both conics are nondegenerate, they are equivalent to $XZ = Y^2$ or $(U, V) \mapsto (U^2, UV, V^2)$ which are degree 2. Thus, by 1.9, they intersect at most $2n = 2(2) = 4$ times but $C_1$ and $C_2$ must intersect at least 5. Thus case 1 is impossible.

*Case 2:* One conic is degenerate and the other is nondegenerate.
WLOG: Assume $C_1$ is the degenerate conic. $C_1$ will be either a line or a line pair while $C_2$ is a non-degenerate conic. By 1.9, $C_2$ will intersect with each line of $C_1$ at most 2 times. Thus, $C_1$ and $C_2$ intersect at most $2+2 = 4$ times but they must intersect at least 5 times. Thus case 2 is impossible.

*Case 3a:* Both degenerate, don't share a line.
$C_1$ and $C_2$ will either be lines or line pairs. Each line of $C_1$ will intersect with each line of $C_2$ at most 1 time. Thus $C_1$ and $C_2$ will intersect at most $1+1+1+1 = 4$ times but they must intersect at least 5 times. Thus case 3a is impossible.

*Case 3b:* Both degenerate, share a line.
Since they share a line, we can write $C_1 = L_0 \cup L_1$ and $C_2 = L_0 \cup L_2$. Thus $\{\heartsuit_1, \heartsuit_2, \heartsuit_3, \heartsuit_4, \heartsuit_5\} \subset C_1 \cap C_2 = L_0 \cup (L_1 \cap L_2)$. But note that $L_1 \cap L_2$ can only contain 1 point. Thus, the other four points must lie on the line $L_0$ but they can't because no four points are collinear. Therefore case 3b is impossible.

Since each case is impossible, our assumption that $C_1 \neq C_2$ must have been wrong. ■

Lets define $S_2 = \{\text{Quadratic forms in } \mathbb{R}^3\} = \{3 \text{ by } 3 \text{ symmetric matrices}\} \cong \mathbb{R}^6$.

Lets fix $\heartsuit_0 = (X_0, Y_0, Z_0) \in \mathbb{P}^2(\mathbb{R})$. Now, we can define $S_2(\heartsuit_0) = \{Q \in S_2 \text{ such that } Q(\heartsuit_0) = 0\}$. For any $Q \in S_2(\heartsuit_0)$ we can write $Q(X_0, Y_0, Z_0) = aX_0^2 + bX_0Y_0 + \cdots + fZ_0^2 = 0$ which is a single linear equation with 6 variables: a,b,c,d,e,f. Thus, $\dim S_2(\heartsuit_0) = 5$.

Similarly, lets fix $\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots \heartsuit_n \in \mathbb{P}^2(\mathbb{R})$ and define $S_2(\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots \heartsuit_n) = \{Q \in S_2$ such that $Q(\heartsuit_i) = 0$ for $i = 1, 2, 3, \cdots n\}$. Instead a single linear equation, this gives us n linear equations with 6 variables each. Thus, $\dim S_2(\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots, \heartsuit_n) \geq 6 - n$.

**Corollary (1.11):**
If $n \leq 5$ and no 4 points are collinear then $\dim(\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots, \heartsuit_n) = 6 - n$.

**Proof:**
We will first show that $\dim(\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots, \heartsuit_n) \leq 6 - n$

*Case $n = 5$:*
$\dim S_2(\heartsuit_1, \heartsuit_2, \heartsuit_3, \heartsuit_4, \heartsuit_5) \leq 1 = 6 - 5 = 6 - n$. (by 1.10)

*Case $n \leq 4$:*
Pick 5-n points so that no 4 are collinear. This will give us a total of 5 points. Thus $1 = \dim(\heartsuit_1, \heartsuit_2, \heartsuit_3, \heartsuit_4, \heartsuit_5) \geq \dim S_2(\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots, \heartsuit_5) - (5 - n)$. Adding $5 - n$ to both sides, we get $6 - n \geq \dim S_2(\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots, \heartsuit_5)$.

We combine this with our previous result that $6 - n \leq \dim S_2(\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots, \heartsuit_5)$ to get $6 - n = \dim S_2(\heartsuit_1, \heartsuit_2, \heartsuit_3, \cdots, \heartsuit_5)$. ■

Fernando's Part

Finding Tangent Lines in Affine and Projective Spaces

**Affine** Lets say we have a curve $f(x,y)=0$.
We want to find the tangent line at a point $\heartsuit = (x_0, y_0)$ on the curve.
Thankfully, $\nabla f(x_0,y_0) = (\frac{\delta f}{\delta x}(x_0,y_0), \frac{\delta f}{\delta y}(x_0,y_0))$ will always be tangent to our curve.
Thus, if $(x,y)$ is on the targent line, then $\nabla f \cdot (x-x_0, y-y_0) = 0$.
So $\frac{\delta f}{\delta x}(\heartsuit)(x-x_0) + \frac{\delta f}{\delta y}(\heartsuit)(y-y_0) = 0$.
Thus our tangent line can be written as $\frac{\delta f}{\delta x}(\heartsuit)x + \frac{\delta f}{\delta y}(\heartsuit)y + (\frac{\delta f}{\delta x}(\heartsuit)x_0 + \frac{\delta f}{\delta y}(\heartsuit)y_0) = 0$.

**Projective** We can naively convert this equation as follows:
$\frac{\delta f}{\delta x}(\heartsuit)X + \frac{\delta f}{\delta y}(\heartsuit)Y + (\frac{\delta f}{\delta x}(\heartsuit)x_0 + \frac{\delta f}{\delta y}(\heartsuit)y_0)Z = 0$
The problem is that this equation includes $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$ which refer to f, which isn't the projective equation.

So we let $F(X,Y,Z) = Z^d f(\frac{X}{Z}, \frac{Y}{Z})$ where d is the degree of f.
Now, we calculate that $F_X = Z^d(\frac{\delta f}{\delta x}\frac{1}{Z} + \frac{\delta f}{\delta y}0) = Z^{d-1}\frac{\delta f}{\delta x}(\frac{X}{Z}, \frac{Y}{Z})$
Similarly, $F_Y = Z^{d-1}\frac{\delta f}{\delta y}(\frac{X}{Z}, \frac{Y}{Z})$.
Finally, we calculate that $F_Z = dZ^{d-1}f(\frac{X}{Z}, \frac{Y}{Z}) + Z^d(-\frac{\delta f}{\delta x}(\frac{X}{Z}, \frac{Y}{Z})\frac{X}{Z^2} - \frac{\delta f}{\delta y}(\frac{X}{Z}, \frac{Y}{Z})\frac{Y}{Z^2})$.

$F_Z$ looks complicated until we plug in a point $\heartsuit = [X_0 : Y_0 : Z_0]$ on our curve.
Since $\heartsuit$ is on our curve, $f(X_0/Z_0, Y_0/Z_0) = 0$. Thus,

$$F_X(\heartsuit) = Z_0^{d-1}\frac{\delta f}{\delta x}(\heartsuit)$$

$$F_Y(\heartsuit) = Z_0^{d-1}\frac{\delta f}{\delta y}(\heartsuit)$$

$$F_Z = Z_0^{d-1}\left(-\frac{\delta f}{\delta x}(\heartsuit)\frac{X_0}{Z_0} - \frac{\delta f}{\delta y}(\heartsuit)\frac{Y_0}{Z_0}\right)$$

Substituting these into our naive equation, we get:
$\frac{1}{Z_0^{d-1}}F_X(\heartsuit)X + \frac{1}{Z_0^{d-1}}F_Y(\heartsuit)Y + \frac{1}{Z_0^{d-1}}F_Z(\heartsuit)Z = 0$.
Finally, we multiply everything by $Z_0^{d-1}$ to get the following elegant projective tangent line equation: $F_X(\heartsuit)X + F_Y(\heartsuit)Y + F_Z(\heartsuit)Z = 0$.