**Algebraic Geometry: Feb. 12**
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To start, we thought back to the last class when we discussed what the conics might look like in projective space. Similar to $\mathbb{R}^2$, there are a limited number of conics in projective space of $\mathbb{R}$, $\mathbb{P}^2(\mathbb{R})$, which are the following:
$X^2 + Y^2 + Z^2 = 0$ which are the empty set
$X^2 + Y^2 - Z^2 = 0$ the non-degenerate conics (ellipses, hyperbola, parabolas)
$X^2 + Y^2 = 0$ the point [0:0:1]
$X^2 - Y^2 = 0$ two lines
$X^2 = 0$ one line
These are all possible conics in $\mathbb{P}^2(\mathbb{R})$ up to a change in variable. Now we will take a look more at conics in $\mathbb{P}^2(\mathbb{R})$. We want to show that the conics are rational curves in projective space, meaning that we can parameterize them by a polynomial/rational function of the line.

## Conic roots are rational:

Looking at the non-degenerate conics, $X^2 + Y^2 - Z^2$ or $X^2 + Y^2 = Z^2$, we showed that in $\mathbb{R}$ we could parameterize the affine part $x^2 + y^2 = 1$ as $t \mapsto ((1-t^2)/(1+t^2), (2t)/(1+t))$.

We now want a projective parameterization by making it a triple and scaling it. A mapping which takes the variable $t$ to projective space $t \mapsto [1 + t^2 : 2t : 1 + t^2]$ does not quite work because $t$ is affine so it does not include the point $[-1 : 0 : 1]$. This means we want to use a mapping from the projective line for this parameterization. What we want to do is take two coordinates $[T : S]$, which should be equivalent when we scale and we get the parameterization

$$[T : S] \mapsto [S^2 - T^2 : 2TS : S^2 + T^2]$$

which will be well defined when dealing with homogeneous polynomial. This parameterization now includes the previously missing point, as [1: 0] maps to [-1; 0; 1].

Although this parameterization works, the parameterization which Miles Reid uses in his book is different and involves a change of variables. In his version we have: $X_1 = X + Z, Y_1 = -X + Z, Z_1 = Y$ and our new equation gives us $-X_1Y_1 + Z_1^2 = 0$ which we can now rewrite as $X_1Y_1 = Z_1$. Reid then wants us to swap $Y_1$ and $Z_1$ to get $XZ = Y^2$.

From this, Reid gets the parameterization

$$[U : V] \mapsto [U^2 : UV : V^2]$$

and now we have found all monomials of degree 2 in one triple. We have embedded the line into $\mathbb{P}(\mathbb{R})$ which will help us when intersecting conics.

Notice that $[U : V]$ is in $\mathbb{P}^1$ and $[U^2 : UV : V^2]$ is a non-degenerate conic in $\mathbb{P}^2$. The motivation behind this is if we want to intersect conics with curves we want to plug in equations to find U and V.

Now we want to think about the **forms of degree d in two variables**. Here, "form" means a homogeneous polynomial, i.e., something that looks like

$$F(U, V) = a_dU^d + a_{d-1}U^{d-1}V + ... + a_1UV^{d-1} + a_0V^d$$

where $a_j$ are not all 0.

Going back to affine space where V=1 we get

$$f(u) = a_d u^d + ... + a_1 u + a_0$$

which may not end up being degree d depending on which $a_j$ from the polynomial above were 0. We want to find the roots of this polynomial. We have that if $f(\alpha) = 0$ then $F(\alpha, 1) = 0$. So if $\alpha = 0$, then we can factor out to $f(u) = (u - \alpha)g(u)$ which is equivalent to $F(U, V) = (U - \alpha V)G(U, V)$. The roots still give factors, but $F(U, V)$ has an extra root where $V = 0$. Then we have that $F(1, 0) = 0$ is equivalent to $ad = 0$ which is equivalent to $F(U, V) = VG(U, V)$, so if a polynomial has enough roots and enough V here, we can completely factor out our polynomial into degree one terms.

## Multiplicity of a root

Next, we want to define the multiplicity of a root. If $F(\alpha, 1) = 0$, the multiplicity of $\alpha$, $m(\alpha)$ is the highest power of $(U - \alpha V)$ that we can factor out. For example, when $F(1, 0)$, $m([1 : 0])$ is the highest power of V which is equal to d- degree of $f(u)$.

This gives us the result that the number of roots of $F(U, V) = 0$ counted with multiplicity must be less than d (over an algebraically closed field, it is equal to d). Even on the projective line, with this form of equation you still get at most d roots.

We can use this to get a count of how many intersections there are between curves. A general theorem, Bezout's theorem, says that given two curves C and D where degree of C is m and degree of D is n, then the number of intersections between these two curves is mn. This theorem is too difficult to prove at this point, but a simpler result is the number of intersections of a curve and a line.

Suppose we have a curve $C = F(X, Y, Z)$ of degree n in $\mathbb{P}^2$ and a line in $\mathbb{P}^2$ given by $L : \alpha x + \beta y + \gamma z = 0 = \{[X(U, V) : Y(U, V) : Z(U, V)]\}$. Plug $X = X(U, V), Y = Y(U, V), Z + Z(U, V)$ into $F(X, Y, Z)$ to get $G(U, V) = 0$. What is the degree of G? Plugging a degree 1 equation into a monomial makes G homogeneous of degree 1. This means there are at most n roots and at most n intersections (exactly n over algebraically closed fields). This means that we can now connect the algebra and geometry of these curves. If we want to see geometrically if a curve has a certain degree we can choose any line and look at how many time it intersects a curve. In an algebraically closed field, the number of intersections will be exactly the degree of the curve.

Another question we can ask is what happens when we intersect a curve with a (non-degenerate) conic Q? When we parameterize $Q = [U^2 : UV : V^2]$ and again plug in, we get an equation $G(U, V) = 0$ of degree 2n, so there will be at most 2n intersections (exactly 2n for algebraically closed fields).

What makes Bezout's theorem so difficult is that this result is dependent on being able to parameterize lines and non-degenerate conics, but most curves cannot be parameterized. So, another question to ask is what happens when we intersect a curve with degenerate conics. The empty conic and single point conic are not interesting, but what is is the conic which is 2 lines. We can factor out this conic into $(X + Y)(X - Y) = 0$, but it is still difficult to parameterize these kinds of curves. There is also a possibility that in the degenerate case, two curves can share entire sections which is infinitely many points. This means if we want to look at the case of degenerate conics, we want to look at those cases where they do not have entire components in common.

This whole idea of being able to find an exact number of intersections over algebraically closed fields inspired mathematicians to think of $\mathbb{C}$ as the place where they should be looking at these curves. There are issues with looking at $\mathbb{C}$ though because many things don't look nice in $\mathbb{C}$ For example, the projective plane in $\mathbb{R}$ is a circle, but in $\mathbb{C}$ it is a sphere. This leads to the question which is: what would complex conics look like in projective space?