The Projective Plane and Line at Infinity: Feb. 10

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Today's lecture details chapters 1.4-1.6 in Reid's book, and is all about the projective plane and what is known as the "line at infinity". It also details how the conic sections are actually all the same shape in the projective plane.

Definition (Projective Plane): Beginning with $\mathbb{R}^3$, remove the origin and setup and equivalence relation defined by (x,y,z)~(λx,λy,λz) for any λ≠0. We denote this equivalence as [x:y:z], representing the ratios between x,y, and z. This forms a projective plane, which we denote as $\mathbb{P}^2$.

There are two key advantages of working in the projective plane. The first is that it removes some exceptions of working in $\mathbb{R}^3$. Most notably, all lines in the projective plane intersect at some point. The second advantage is that the projective plane is bounded.

Definition (Line at Infinity): The set of points in the equivalence class: { [x:y:0] } is defined to be the line at infinity.

A line passing through the line at infinity has equation Ax+By=0. In $\mathbb{R}^3$, this line has equation Ax+By+C=0, so all lines of the same slope pass through the same line at infinity. As a result, lines in $\mathbb{R}^3$ that are parallel intersect at the line at infinity in $\mathbb{P}^2$.

Definition (Homogenous Polynomial): A polynomial is said to be homogenous if all of its monomials are of the same degree.

While working in $\mathbb{P}^2$, only equations of homogeneous polynomials that equal zero make sense to work with. To determine where two lines intersect, we begin with their equations in $\mathbb{R}^2$, say y=mx+b and y=mx+d. Converting these equations to coordinates in $\mathbb{P}^2$ gives $\frac{y}{z} = m\frac{x}{z} + b$ and $\frac{y}{z} = m\frac{x}{z} + d$. To convert these equations to homogenous polynomials, we multiply through by z and set them equal to zero, resulting in y-mx-bz=0 and y-mx-dz=0. If we want to determine where these lines intersect at infinity, we know that z=0 there, and since the ratio between the y and x coefficients is 1:m, the equation of the line at infinity at which these lines intersect is [1:m:0].

Definition (Projectivity): A projectivity of $\mathbb{P}^2$ is a function T(x)=Mx where M is a 3x3 invertible matrix.

We are able to use projectivities and orthonormalization to ensure that any conic section in $\mathbb{P}^2$ can be written in the form [x y z]M $[x\ y\ z]^T$ where M is a diagonal 3x3 matrix. After changing variables, any conic section in $\mathbb{P}^2$ can be written as $\boldsymbol{\alpha} x^2 + \boldsymbol{\beta} y^2 + \boldsymbol{\gamma} z^2 = 0$ where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \{1, -1, 0\}$. When they are written like this, they have the following properties based on their coefficients:

$x^2 + y^2 + z^2 = 0$	Empty
$x^2 + y^2 - z^2 = 0$	Non degenerate conic
$x^2 + y^2 = 0$	Point
$x^2 - y^2 = 0$	Two lines
$x^2 = 0$	One line

An ellipse in $\mathbb{P}^2$ does not intersect the line at infinity. A parabola in $\mathbb{P}^2$ intersects the line at infinity once, while a hyperbola in $\mathbb{P}^2$ intersects the line at infinity twice. Topologically, all conics in $\mathbb{P}^2$ are circles since the parabola and hyperbola intersect at the line at infinity.