Algebraic geometry: Feb. 7

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Today's lecture covers essentially the first three sections in Reid's book. We will consider some of the simplest algebraic curves. The simplest of all is a line. The first question we need to address is that where should our line live? For now, we want to consider lines in the affine plane.

Definition 1. *Let k be field. Then the affine plane over k, denoted as $\mathbb{A}^2$, is the set $\{(x, y) : x, y \in k\}$.*

For our purpose today, k will always be $\mathbb{R}$. The set of $\mathbb{R}$-points in $\mathbb{A}^2$ in denoted as $\mathbb{A}^2(\mathbb{R})$, and this is merely another way of saying $\mathbb{R}^2$. A line in $\mathbb{A}^2$ is defined by an equation of the form

$$ax + by + c = 0, \qquad a, b, c \in \mathbb{R} \text{ and } a, b \text{ not both } 0.$$

How do we check that this equation actually define a line? The idea is that a line should look like an axis. So consider a parameter t and set $y = t$ (this assumes that $a \neq 0$; if $a = 0$, then instead let $x = t$). Solving for x, we get

$$x = -\frac{1}{a(bt + x)}.$$

Therefore, every points on the curve traced out by this equation can be described by the following pair:

$$(x, y) = \left(-\frac{1}{a(bt + c)}, t\right).$$

There are two ways of interpreting this pair of coordinates. The first interpretation is that this gives a mapping from the t-axis to the line. In fact, it's easy to see that it's an isomorphism with inverse given by $(a, b) \mapsto b$. Another way to look at this situation is that this line can be viewed as a point in the affine plane over the function field $\mathbb{R}(t)$.

It's also worth noticing that the equation defining a line need not to be of degree 1; we can simply take the original equation and take it to n-th power to get a new equation. But the set of zeros of the new equations has to be the same as the set of zeros of the original equation. Thus, they trace out the same line. A perhaps more interesting way is that we could multiply the original polynomial by some polynomial with no real roots. The product clearly will trace out the same line, and the degree of the new polynomial is greater than 1. This is a delicate issue that we will explore more in the future.

The next simplest curve is a circle. Consider the circle given by the equation

$$x^2 + y^2 = 1.$$

This circle has a rational point $(-1, 0)$. Now we can start to draw lines from $(-1, 0)$ that intersect the circle. They will also intersect y-axis at some point. Call the point $(0, t)$. The point this line intersect with the circle will be a function of t, so we can call it $x(t), y(t)$. To see what the coordinates actually are, we can work out the algebra: the equation of the line is $y = tx + t$, and to find $(x(t), y(t))$, we simply plug it back into the circle equation to get

$$\begin{aligned} x^2 + (tx + t)^2 &= 1 \\ x^2 + t^2x^2 + 2t^2x + t^2 - 1 &= 0 \\ (1 + t^2)x^2 + 2t^2x + t^2 - 1 &= 0. \end{aligned}$$

Thus, we need to find the roots of some quadratic polynomial. We already know one of the roots: since the line also intersects the circle at the point $(-1,0)$, then one of the roots has to be $x=-1$. Now Vieta's formula tells us that product of two roots is given by c/a, so $-x(t)=\frac{t^2-1}{1+t^2}$. Then we have

$$(x(t),y(t))=\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right).$$

Now it seems that we again have a function from the t-axis to the circle. But note that the map is

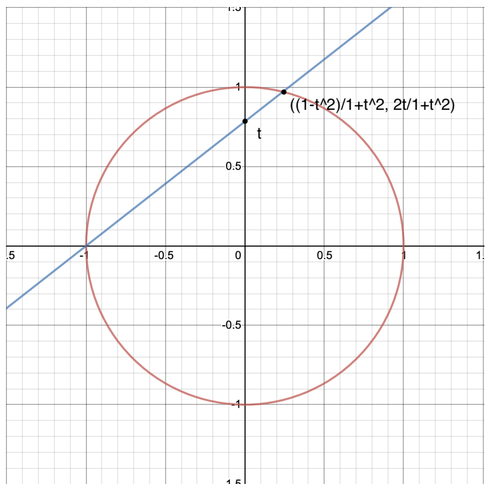

Figure 1: Parametrizing a circle.

not onto (it's one-to-one since $t=\frac{y(t)}{1+x(t)}$): the point $(-1,0)$ is not in the image of this map. This is expected, if the line intersects at $(-1,0)$ twice, then we would expect the line to be vertical, or in some sense, to have infinite slope. Therefore, we really want to map ∞ to $(-1,0)$. Moreover, if we let $t\to\infty$, we get

$$\lim_{t\to\infty}(x(t),y(t))=\lim_{t\to\infty}\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right)=(-1,0).$$

This matches well with out intuition. Another issue is that for some field (such as $\mathbb{C}$) we can have $1+t^2=0$, which means that the function is not defined at some points on the t-axis. However, this parametrization says that a circle is in fact pretty close to be a line. Moreover, observe that the parametrization is defined over $\mathbb{Q}$, which means plugging any rational number t will produce a rational point on the circle. Since finding rational points on the circle is equivalent to finding Pythagorean triples, this parametrization offers us a way to find all Pythagorean triples. We will make a new definition.

Definition 2. *If a curve can be parameterized by rational functions (i.e. ratios of polynomials), we call it a rational curve.*

In particular, our circle is in fact a rational curve. Now in general, if we replace the equation with any degree 2 equation, fix a point on the curve and start to draw lines through it, we can use a similar method to parametrize this general curve. One would expect this method to work since plugging in a degree 1 equation of a line into the degree two equation will produce again a quadratic equation and everything should proceed in a similar way. So, unless the case is really strange, we expect to be able to parametrize curves traced our by degree 2 polynomials.

As an example, consider the curved traced out by

$$2x^2 + y^2 = 5.$$

It's an ellipse, and it contains the point $(0, \sqrt{5})$. Using this point, we indeed get a parametrization

$$(x(t), y(t)) = \left(\frac{10t}{5 + 2t^2}, \frac{\sqrt{5}(2t^2 - 5)}{5 + 2t^2} \right).$$

Note that this time the rational functions are no longer defined over $\mathbb{Q}$, so we cannot find rational points on this curve via the parametrization. In fact, we don't even know if there's any rational points on the curve by looking at the parametrization. The answer in fact is no, and can be gotten by doing modular arithmetic mod 5: suppose $x = a/c$ and $y = b/c$. Then we have the equation $2a^2 + b^2 = 5c^2$. We can assume that a, b, c have no common factors, but then this forces a and b to be not both divisible by 5. Then all we need to do is to check all the cases in $\mathbb{Z}/5\mathbb{Z}$, and one quickly realizes that this equation has no non-zero solution $\mod 5$. Then it cannot have any non-trivial solution in $\mathbb{Z}$ to begin with. The interesting (and very hard) number theory question is that if we have non-trivial solutions modulo every prime, are we guaranteed to find a non-trivial solution for the original equation. But let's remind ourselves that this is a geometry class and we shall resist the temptation to go any further on this topic.

Finally, let's consider the conics. Let

$$q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f.$$

What are some of the possibilities? We can get

1. A parabola (e.g. $x^2 - y = 0$);

2. a circle (e.g. $x^2 + y^2 = 1$) or an ellipse (e.g. $2x^2 + y^2 = 1$);

3. a hyperbola (e.g. $x^2 - y^2 = 1$);

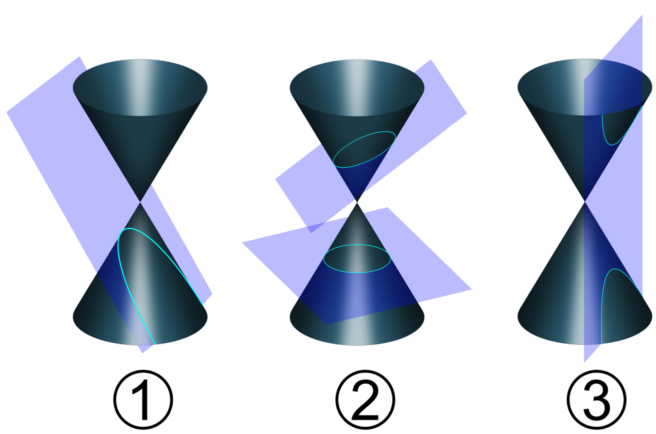

Figure 2: Non-degenerate Conics. Picture credit to Pbroks13 from Wikipedia.

Of course, there are always degenerate cases:

1. A single point (e.g. $x^2 + y^2 = 0$);

2. a single line (e.g. $(x-y)^2 = 0$);
3. two lines (e.g. $x^2 - y^2 = 0$);
4. no solutions (e.g. $x^2 + y^2 + 1 = 0$);
5. the entire plane (e.g. $0 = 0$)

The claim is that these all are the cases. There are at least two ways of proving this: the brute force method and a linear algebra method.

The brute force method. Suppose $a \neq 0$. We can first divide through by a to get

$$x^2 + \frac{b}{a}xy + \frac{c}{a}y^2 + \frac{d}{a}x + \frac{e}{a}y + \frac{f}{a} = 0.$$

The it's just a matter of completing squares:

$$\begin{aligned} x^2 + \frac{b}{a}xy + \frac{c}{a}y^2 + \frac{d}{a}x + \frac{e}{a}y + \frac{f}{a} &= 0 \\ x^2 + \frac{b}{a}xy + \frac{b^2}{4a^2}y^2 + \frac{c}{a}y^2 + \frac{d}{a}x + \frac{e}{a}y + \frac{f}{a} &= 0 \\ (x + \frac{b}{2a}y)^2 + (\frac{c}{a} - \frac{b^2}{4a^2})y^2 + \frac{d}{a}(x + \frac{b}{2a}) + (\frac{e}{a} - \frac{db}{2a^2})y + \frac{f}{a} &= 0. \end{aligned}$$

Setting $X = x + \frac{b}{2a}y$, we got a new equation of the form

$$X^2 + By^2 + DX + Ey + F = 0.$$

Now if $B = 0$, we are done: if $F \neq 0$, we get a parabola, and if $F = 0$, we get some degeneracy. If $B \neq 0$, then completing the square again, we can get it into the form

$$\alpha u^2 + \beta v^2 + c = 0.$$

From here we can get the full classification. This is quite unpleasant and not very insightful.

The linear algebra method. We first observe that

$$ax^2 + bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the original equation can be written as

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + f = 0.$$

A theorem from linear algebra says that any symmetric matrix can be diagonalized via orthogonal matrices; in other words, $\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} = S^{-1}DS$, where D is a diagonal matrix of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $S^{-1} = S^T$. This change of basis amounts to a change of variable and we end up with

$$AX^2 + BY^2 + \text{stuff} = 0,$$

where stuff is of degree 1. From here one can also complete the classification of conics. In fact, one can show that the determinant of the matrix is what determines which conic section this equation produces (as long as you are not in the degenerate case).

If we work over the projective plane, the non-degenerate conics are all the same. One could then think about all the conics, and one see that these conics can be specified by the coefficients of the equations up to some scaling. Then all the conics can be viewed as some subsets of higher dimensional spaces, and one could ask interesting questions there.

The class ends with a challenge. Consider the rational curve parametrized by $t \mapsto (t, t^2, t^3)$ in $\mathbb{A}^3$. This is called the twisted cubic. It's in fact non-planar. The challenge is to find the polynomials that cut out this curve.