MA434, Spring 2020 — Midterm

This is a take-home test. It is due by 5PM on March 13. That is a firm deadline.

The test contains 12 questions, but you don't need to solve all of them. The number of points for each question is indicated (most are worth 20 points). The problems are *not* arranged in order of difficulty.

You should solve enough problems to get at least 100 points, but the total value of the problems (full problems, not fractions of problems) you turn in can be no more than 120 points. The maximum score you can get is 100, so doing an extra 20 points' worth only buys you insurance. When I grade, I will go until I have graded problems worth up to the limit of 120 points, then stop.

Rules of the Game: While you work on this test, you may consult Miles Reid's *Undergraduate Algebraic Geometry*, your class notes and the scibal class notes.. You may use *Sage* or GP or other mathematical software to do computations. You may use an Abstract Algebra textbook for reference; if you do, please tell me what book you have used. No other reference materials are allowed. In particular, you should not search the internet for solutions.

You should work entirely by yourself. You may talk to me (though I don't promise to answer every question you might have), but you may not talk to anyone else. (Complaining and asking for sympathy are ok, but don't discuss the actual content of the test.)

Finally, you should really *write up* (and not just write down) your solutions: they should read as if they were an example in a well-written textbook. Make sure that you explain carefully your line of reasoning at each point; your text should be such that another student at about your level could follow the steps without having to ask you for help. To achieve this goal, be as verbose as necessary — it is better to write too much than too little. Solutions should be written as carefully and legibly as possible. (The professor has old eyes.) *Don't turn in first-draft material.*

Together with your test, you should turn in a signed statement saying that you have followed the test rules as stated above.

Good luck!

1. [20 points] Suppose $f(X,Y,Z)$ is a homogeneous polynomial of degree n with coefficients in $\mathbb{R}$, so that we have $f(tX,tY,tZ) = t^n f(X,Y,Z)$. Show that

$$X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} = nf.$$

(Hint: this is true for any differentiable function that satisfies the equation $f(tX,tY,tZ) = t^n f(X,Y,Z)$, not just for polynomials; use calculus.)

It's worth pointing out that this shows that if a point satisfies

$$\frac{\partial f}{\partial X}(P) = \frac{\partial f}{\partial Y}(P) = \frac{\partial f}{\partial Z}(P) = 0,$$

then P is automatically on the curve defined by $f(X,Y,Z) = 0$.

2. [20 points] The Proposition in section 1.13 of *Undergraduate Algebraic Geometry* says that in a pencil of conics *containing at least one non-degenerate conic* there will be at most 3 degenerate conics, and if $k = \mathbb{R}$ there will always be at least one degenerate conic. Find an example of a pencil of conics over $\mathbb{R}$ that does not contain any non-degenerate conics.

3. [30 points] The Proposition in section 1.13 of *Undergraduate Algebraic Geometry* says that in a pencil of conics containing at least one non-degenerate conic there will be at most 3 degenerate conics, and if $k = \mathbb{R}$ *there will always be at least one degenerate conic*. In the examples we played with there were always at least two degenerate conics. Find an example with exactly one.

4. [20 points] Let $f(x,y)$ be a polynomial in two variables with coefficients in $\mathbb{R}$ such that $f(0,0) = 0$. We want to look at the point $P = (0,0)$ on the affine curve C defined by $f(x,y) = 0$.

a. The line $y = 0$ passes through P. Under what condition on f is it the tangent line to C at P?

b. Under what conditions on f is the point P an inflection point with tangent line $y = 0$?

Can you generalize to other lines? Other points?

5. [20 points] This problem describes another way of thinking about the projective line $\mathbb{P}^1(k)$. Remember that the affine line $\mathbb{A}^1(k)$ is just another name for the field k.

Any point in $\mathbb{P}^1(k)$ looks like $[u : v]$ with $u, v \in k$. Define the subsets

$$U = \{[u : v] \in \mathbb{P}^1(k) \mid v \neq 0\}$$

and

$$V = \{[u : v] \in \mathbb{P}^1(k) \mid u \neq 0\}.$$

a. If $[u, v] \in U$, define $f([u : v]) = u/v$. Show that f is a bijection between U and $\mathbb{A}^1(k)$.

b. If $[u, v] \in V$, define $g([u : v]) = v/u$. Show that g is a bijection between V and $\mathbb{A}^1(k)$.

c. Suppose $t \in \mathbb{A}^1(k)$, $t \neq 0$. What is $f(g^{-1}(t))$?

d. Explain how this means that we can think of $\mathbb{P}^1(k)$ as the result of gluing two copies of $\mathbb{A}^1(k)$ along the subsets $A^1(k) - \{0\}$ via the function $t \mapsto 1/t$. (If you prefer to avoid the language of "gluing." you can express it as taking the disjoint union of two copies of $A^1(k)$ and then passing to the quotient with respect to an equivalence relation.)

6. [20 points] Let E be the cubic in $\mathbb{P}^2(\mathbb{Q})$ defined by the affine equation in Weierstrass form

$$y^2 = x^3 + x + 1.$$

The point $P = (0, 1)$ is on E. Use the group law to compute $2P$, $3P$, and $4P$. (The numbers will get ugly, so use software. It's ok to use *Sage*'s built-in functions if you can figure out how to do it.)

7. [20 points] The curve $y^2 = x^3 + x + 1$ also makes sense over $\mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$. Of course, in that case there will be finitely many points on the curve.

a. Find all the points of E in $\mathbb{P}^2(\mathbb{F}_7)$. (Don't forget the point at infinity.)

b. The set of all points with coordinates in $\mathbb{F}_7$ is a group. What group is it?

8. [20 points] (Gauss's Lemma) Suppose R is a UFD and K is its field of fractions. We want to compare factorizations in $R[x]$ and in $K[x]$. Let $f(x) \in R[x]$ and suppose we have $g(x), h(x) \in K[x]$ such that $f(x) = g(x)h(x)$. Show that there exists $a \in K$ such that $\tilde{g}(x) = ag(x) \in R[x]$, $\tilde{h}(x) = \frac{1}{a}h(x) \in R[x]$, and so $f(x) = \tilde{g}(x)\tilde{h}(x)$ is a factorization in $R[x]$.

(It's useful to remember that in a UFD every irreducible element is prime and that if D is a domain so is $D[x]$.)

9. [30 points] Let V be the variety in $\mathbb{A}^3(k)$ defined by the equations $xy = 0$, $xz = 0$, $yz = 0$ (points on V are those for which all three equations hold).

a. Describe V. In particular, explain why it is a curve in $\mathbb{A}^3(k)$.

b. Show that the set

$$I = \{f \in k[x, y, z] \mid f(P) = 0 \text{ for all } P \in V\}$$

of polynomials that vanish on V is an ideal in $k[x, y, z]$.

c. Show that I is generated by xy, xz, yz.

d. Intuitively, we would expect that a curve in $\mathbb{A}^3(k)$ is determined by two polynomial equations. Do there exist polynomials f and g such that I is generated by f and g?

10. [20 points] Let C be the curve in $\mathbb{P}^2$ whose affine equation is $y^2 = x^3 + x^2$. This is the nodal cubic we studied in section 2.1. Show that the line $y = tx$ has a double intersection with C at $(0, 0)$ and find the third point of intersection. Check that this gives the parametrization in 2.1. What happens when $t = \pm 1$?

11. [20 points] With C as in the previous problem, let $C(k)$ be the set of points on C with coefficients in k (including the point at infinity), and let $C'(k) = C(k) - \{(0, 0)\}$. (So $C'(k)$ is the set of points on C where there is a unique tangent.) We want to try to define a group structure using the same method as for nonsingular cubics.

a. Let A be a point in $C(k)$ and let $P = (0, 0)$. Let L be the line through A and P. What is the third intersection of L and C?

b. Explain why the point P is problematic if we want a group structure.

c. Suppose $A, B \in C'(k)$, and let L be the line through A and B. Show that the third intersection of L with C is in $C'(k)$.

d. Explain why this gives a group law on $C'(k)$.

(It turns out that with this group law $C'(k) \cong k^\times$, but this is a little hard to prove.)

12. [30 points] Suppose $f, g \in k[x, y]$ are polynomials in two variables with coefficients in a field k. Suppose $f(x, y)$ is irreducible and does not divide $g(x, y)$. Show that there are at most finitely many solutions to $f(x, y) = g(x, y) = 0$.

(Hints: since $f(x, y)$ is not a constant, we may suppose without loss of generality that x appears in $f(x, y)$ with positive degree. Move from $k[x, y]$ to $k(y)[x]$, which is a ring of polynomials over a field. Check that $f(x, y)$ is still irreducible, that it still does not divide $g(x, y)$, and therefore that $\gcd(f, g) = 1$. Since $k(y)[x]$ is a PID, it follows that we can write 1 as a linear combination of f and g. Go on from there.)