**Proposition 2.** If \( \lim_{z \to z_0} f(z) = a \) and \( \lim_{z \to z_0} g(z) = b \), then

i) \( \lim_{z \to z_0} (f + g) = a + b \)

ii) \( \lim_{z \to z_0} (fg) = ab \)

iii) \( \lim_{z \to z_0} \frac{f}{g} = \frac{a}{b} \) if \( b \neq 0 \).

Proof is identical for limits of real numbers. See notes for proof of (ii).

**Proof of (ii)**

\[
|f(z)g(z) - ab| = |f(z)g(z) - ag(z) + ag(z) - ab| \\
\leq |g(z)||f(z) - a| + |a||g(z) - b|
\]

Now \( \forall \varepsilon > 0 \exists \delta > 0 \) such that for \( |z - z_0| < \min(1, \delta) \)

\[
|g(z) - b| < \min\left(\frac{\varepsilon}{|a||g(z_0)|}, \frac{|a||g(z_0)|}{\varepsilon}\right) \text{ and } |f(z) - a| < \frac{\varepsilon}{2(1+|b|)}
\]

\[
\Rightarrow |g(z)| < 1 + |b|
\]

So

\[
|f(z)g(z) - ab| < |g(z)||f(z) - a| + |a||g(z) - b| < (1+|b|)\left(\frac{\varepsilon}{2(1+|b|)}\right) + |a|\left(\frac{\varepsilon}{2|a|}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

So \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( |z - z_0| < \delta \) when \( |z - z_0| < \delta \).

**Continuity:** If \( f: \mathbb{C} \to \mathbb{C} \) is defined on an open disk \( D(z_0) \), we say \( f \) is **continuous at** \( z_0 \) if

\[
\lim_{z \to z_0} f(z) = f(z_0)
\]

The limit exists and is the value of the function.
TRANSLATED INTO \( \varepsilon, \delta \) LANGUAGE:

\[ f \text{ is continuous at } z_0 \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \]
\[ |f(z) - f(z_0)| < \varepsilon \text{ if } |z - z_0| < \delta \]

IF \( f \) IS DEFINED ON AN OPEN SET \( S \), WE SAY \( f \) IS CONTINUOUS ON \( S \) IF \( f \) IS CONTINUOUS AT \( z_0 \) FOR ALL \( z_0 \in S \).

WHY MUST \( S \) BE OPEN? SO WE CAN FIND AN OPEN DISK AROUND EACH \( z_0 \in S \).

PROPOSITION 3: IF \( S \) IS AN OPEN SET, \( f, g : S \to \mathbb{C} \) ARE CONTINUOUS ON \( S \) THEN

a) \( f + g \) IS CTS ON \( S \)

b) \( fg \) IS CTS ON \( S \)

c) \( \frac{f}{g} \) IS CTS ON \( S \) WHEN \( g \neq 0 \).

PROOF: Follows from Prop. 2

COROLLARY 1: POLYNOMIAL AND RATIONAL FUNCTIONS ARE CTS.

PROOF: Show \( f_1(z) = z \) and \( f_2(z) = k \) for any \( k \in \mathbb{C} \) ARE CONTINUOUS, USE Prop. 3.
Proposition 4.1: If \( \lim_{z \to z_0} f(z) = a \) and \( g(z) \) is CTS at \( a \), then \( \lim_{z \to z_0} g(f(z)) = g(a) \).

2) \( A, B \) are open sets \( f: A \to C \) and \( g: B \to C \) are CTS and \( f(A) \subset B \), then \( g(f(z)) \) is CTS on \( A \).

**Complex Differentiability**

Let \( S \) be an open set, \( f: S \to C \) a function. We say that \( f \) is **complex differentiable** at \( z_0 \) if the limit \( \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \) exists, \( \lim_{z \to z_0} \frac{f(z + h) - f(z)}{h} \) exists. The limit is denoted by \( f'(z_0) \) or \( \frac{df}{dz}(z_0) \).

\( f \) is said to be **analytic** or **holomorphic** on \( S \) if \( f \) is complex differentiable on each \( z_0 \in S \). We say \( f \) is **entire** if it is differentiable on \( C \).

Is this strange? How do we normally define differentiability from \( \mathbb{R}^2 \to \mathbb{R}^2 \) (more on this soon).

**Proposition 5:** If \( f \) is differentiable at \( z_0 \) then \( f \) is CTS at \( z_0 \).

**Proof:** \( \lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) = f'(z_0) \cdot 0 = 0 \) so \( \lim_{z \to z_0} f(z) = f(z_0) \).
**Ex:** \( f(z) = z \) is entire

**Proof:** \( \forall z_0 \in \mathbb{C}, \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1 \)

\[ f'(z) = 1 \]

**Ex:** For \( \mathbb{C} \) is the constant function \( f(z) = w \)

**Proof:** \( \forall z_0 \in \mathbb{C}, \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{w - w}{z - z_0} = 0 \)

\[ f'(z) = 0 \]

**Example:** \( f(z) = \bar{z} \) is not complex differentiable at any point in \( \mathbb{C} \).

**Proof:** Let \( z_0 \in \mathbb{C} \), \( \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{\bar{z_0 + h} - \bar{z_0}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h} \)

This limit does not exist. Why? Well, let \( h = re^{i\theta}, \quad r > 0, \quad \Theta \in \mathbb{R} \). If \( h \to 0 \) then \( r \to 0 \).

So \( \lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{r \to 0} \frac{re^{-i\theta}}{re^{-i\theta}} = e^{-2i\theta} = \cos 2\theta - i\sin(2\theta) \)

What the heck is \( \Theta \)? It could be any point on the unit circle. Limit does not exist!

More specifically, let \( h(t) = t \) for \( t \in \mathbb{R} \), so \( \lim_{t \to 0} h(t) = 0 \) so \( \lim_{h \to 0} \frac{f(z + h(t)) - f(z)}{h(t)} = \lim_{t \to 0} \frac{t}{t} = 1 \)