People like this formula:

\[ e^{i\theta} + 1 = 0 \]

**Proposition:** With above definition of \( e^z \), \( z = x + iy \)

1. \( e^{u+z} = e^u e^z \)
2. \( e^z \) is never zero
3. \( |e^z| = e^x \)
4. \( \arg(e^z) = \frac{x + 2\pi k}{1} \) \( k \in \mathbb{Z} \)
5. \( f(z) = e^z \) is periodic with period \( 2\pi i \) \( (f(z + 2\pi i)) = f(z) \)
6. \( e^{2\pi ik} = 1 \) for \( k \in \mathbb{Z} \)

**Proof:** You can do it.

What does this map look like?
**Logarithms**

Usually defined as inverse of exponential map, but now $e^z$ is not 1-1.

We make do the best we can.

Let $w 
eq 0$.

\[ \log(w) = \left\{ \ln|w| + i\theta \mid \theta \in \arg(w) \right\} \subseteq \text{INFINITE SET} \]

So it's a set. If $a \in \log(w)$ then

\[ e^a = e^{\ln|w| + i\theta} = e^{\ln|w|} e^{i\theta} = |w|(\cos \theta + i \sin \theta) = w \]

So it's the set of points that exponentiate to $w$.

So $\log(y) = \left\{ \frac{\ln(y) + 2\pi ik}{i} \mid k \in \mathbb{Z} \right\}$

\[ \log(i) = \left\{ \frac{\ln(1i) + i\frac{\pi}{2} + 2\pi ik}{i} \mid k \in \mathbb{Z} \right\} = \left\{ i\left(\frac{1 + 4\pi k}{2}\right) \right\} \]

\[ \log(1 + j3) = \left\{ \ln|1 + j3| + (\frac{\pi}{2} + 2\pi k)i \mid k \in \mathbb{Z} \right\} = \left\{ \ln(2 + (\frac{\pi}{2} + 2\pi k)i) \mid k \in \mathbb{Z} \right\} \]

\[ \tan \theta = j3, \quad \theta = \frac{\pi}{3} \]

\[ 1 + (j3)^2 = -2, \quad |1 + j3| = 2 \]

But a multi-valued log is weird—You

First, let

\[ \text{Log}(w) = \ln|w| + i\text{Arg}(w) \]

\[ \hat{\text{Capital L}} \]

Principal branch of Log.

\[ \text{Log}(y) = \ln|y|, \quad \text{Log}(i) = \frac{i\pi}{2}, \quad \text{Log}(1 + j3) = \ln(2) + \frac{i\pi}{3} \]
WHERE IS IT CONTINUOUS? (WE HAVEN'T DEFINED CONTINUITY YET)

EVERYWHERE BUT HERE, IRRELEVANCE
BAD STUFF HAPPENS HERE
WHY THE θ = π, −π LINE? HAD TO CUT SOMEWHERE!

COMPLEX EXPONENTIALS (AGAIN)

HOW DID WE DEFINE a^z WHEN a ∈ R² AGAIN?

\[ a^z = e^{\ln(a)z} \]

OK SO ON C WE CAN JUST LET

\[ z = \frac{\ln(w)}{\ln(a)} \] POTENTIALLY AN INFINITY OF VALUES

DEFINITION 1

WHAT IS \( i^i \)?

\[ \ln(i) = \frac{\pi}{2} + 2\pi ik \]

\[ i = e^{\frac{\pi}{2} + 2\pi ik} \]

SO \( i^i \) \[ e^{\frac{\pi}{2} + 2\pi ik} \]

\[ \approx \text{DO MANY REAL VALUES} \]

OR WE CAN SPECIFY THE PRINCIPAL BRANCH OF LOG

\[ z^w = e^{z \log(w)} \] DEFINITION 2

IN THIS CASE \( i^i = e^{i \log(i)} = e^{-\pi/2} \)
UNLESS A BRANCH IS SPECIFIED, ASSUME DEF. 1.

ANOTHER EXAMPLE

\[(1 + i \sqrt{3})^{1+i} = ?\]

by \((1 + i \sqrt{3}) = \xi \ln |1| + i(\pi/3 + 2\pi k) \mid k \in \mathbb{Z}\)

so \((1 + i \sqrt{3})^{1+i} = \xi \left( \ln (-\pi/3 + 2\pi k) + i \left( \ln 1 + \pi/3 + 2\pi k \right) \right) \mid k \in \mathbb{Z}\)

so \((1+i \sqrt{3})^{1+i} = \xi e^{-\pi/3 + 2\pi k} e^{i \left( \ln 1 + \frac{\pi}{3} + 2\pi k \right)} \mid k \in \mathbb{Z}\)

\[\text{OR IF WE USE PRINCIPAL BRANCH, } k = 0\]

\[(1 + i \sqrt{3})^{1+i} = 2e^{-\pi/3} \left( \cos \left( \ln(1) + \pi/3 \right) + i \left( \ln(1) + \pi/3 \right) \right)\]

**Remark**: WHAT DOES \(e^z\) MEAN?

Well by \(\xi \ln |z| + i2\pi k \mid k \in \mathbb{Z}\) = \(\xi \left( 1 + 2\pi i k \right) \mid k \in \mathbb{Z}\)

so \(e^z = \xi e^{z + 2\pi i k} \mid k \in \mathbb{Z}\) BY DEF. 1

POTENTIALLY \(\infty\) MANY VALUES

SO WHEN WE SAY \(e^z\) OR \(\exp(z)\)
WE MEAN OUR ORIGINAL DEFINITION FROM SERIES EXPANSION:

\[e^z = e^x (\cos y + i \sin y) \quad z = x + iy\]

OR, IF YOU PREFER, \(e^z\) IS ALWAYS DEFINED BY PRINCIPAL BRANCH
Prop 4.7': (Number of values of $z^w$, $z \neq 0$)

1) If $w \in \mathbb{Z}$, then $z^w$ is single-valued.

2) If $w \in \mathbb{Q}$, $w \neq \mathbb{Z}$ with $w = \frac{p}{q}$ in reduced form, then $z^w$ is $q$-valued.

3) If $w \notin \mathbb{Q}$, $z^w$ is $\infty$-valued.

So why do we need the multivalued version of exponentials?

Again, continuity. We may want to just (come soon!)

Define $3^{\sqrt{3}}$ by the principal branch of log, but then consider $f(z) = z^{\sqrt{3}} = e^{\sqrt{3} \ln |z| + i \sqrt{3} \arg(z)}$

Go around

Start at 3 with $\arg(3) = 0$

End with $\arg(3) = 2\pi$

Continuity of argument forces us to count turns, and forces different values of $3^{\sqrt{3}}$.

But when we start doing calculus on $\mathbb{C}$, we'll want single-valued functions. So we'll restrict the domain of our functions, so we don't encounter these problems.