1. WORKING WITH COMPLEX NUMBERS

- Addition, Subtraction, Multiplication, Division
- Geometric Interpretation, Cartesian vs. Polar x + iy, r e^{i \theta}
- Conjugation, Modulus, \( \overline{z} \overline{z} = |z|^2 \)
- Distance between points |z - w|
- Definition of \( z^n = \frac{e^{n \theta}}{n!} \cdot e^{\lambda (\cos \gamma + i \sin \gamma)} \)
- Definition of trig and hyperbolic trig functions
- De Moivre’s Formula \((\cos \theta + i \sin \theta)^n = \cos n \theta + i \sin n \theta\)
- \( \arg, \Arg, \) other branches of \( \arg \)
- \( \log, \Log, \) other branches of \( \log \)
- Evaluating \( z^w \) as a multi-valued function
- Solving \( z^n = w \)
- Geometry of \( e^w, \log(w) \), visualizing complex maps
- Mobius maps, composition, inverses, preservation of circles and lines, the point at \( \infty \)
- Complex numbers as matrices \( z \mapsto \begin{bmatrix} x & y \\ y & x \end{bmatrix} \)
2. Continuity and Differentiation

- Limits, Continuity and Deriv. of Complex Functions
  - Considering a function \( f: \mathbb{C} \to \mathbb{C} \) as a function \( f: \mathbb{R}^2 \to \mathbb{R}^2 \)
  - Relationship between the \( \mathbb{R}^2 \to \mathbb{R}^2 \) derivative and the \( \mathbb{C} \to \mathbb{C} \) derivative
  - Cauchy-Riemann Theorem (CRT)

- Relation between Holomorphic and Conformal
  - How the matrix interpretation of complex numbers informs the CRT and conformality

\[ \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \]

\[ \nabla^2 u + \frac{\partial^2 v}{\partial y^2} = 0 \]

\[ f \quad \text{geo interp.} \]

\[ f' (z) = 0 \]

\[ \text{Conformal Maps} \]
3. INTEGRATION

- DEFINITION AND BASIC RULES OF CONTOUR INTEGRATION
  \[ \int_C f(z) \, dz = \int_a^b f(z(t)) \, z'(t) \, dt, \quad \text{REPARAMETERIZATION, BREAKING A CURVE UP ETC.} \]

- BOUNDING INTEGRALS
  \[ \left| \int_C f(z) \, dz \right| \leq \max_{y \in [a,b]} |f(y)| \cdot \text{Lגן}, H_{y} \]

- FUNDAMENTAL THM OF COMPLEX CALCULUS
  \[ \int_C f(z) \, dz = F(b) - F(a) \]

- EQUVALENCE THEOREM: PATH INDEPENDENCE = ZERO ON ALL LOOPS = EXISTENCE OF ANTI DERIV.

- HOLONOMIC \rightarrow ANTI-DERIV. ON DISKS

- CAUCHY'S THEOREM (VERSION 3), DEFINITION OF HOMOTOPY ON A DOMAIN, SIMPLY-CONNECTEDNESS

- CAUCHY'S INTEGRAL FORMULA

\[ \Rightarrow \quad \text{CAUCHY-TYPE INTEGRALS ARE HOLONOMIC} \]

\[ \Rightarrow \quad \text{CAUCHY-DIFFERENTIATION FORMULA (CAN INTERCHANGE DERIV. AND INTEGRAL FOR INTEGRALS OF CAUCHY TYPE)} \]

- LOUVILLE'S THEOREM \rightarrow FUNDAMENTAL THM FOR ENTIRE FUNCTIONS OF ALGEBRA

- MAXIMUM MODULUS THEOREM FOR HOLOMORPHIC FUNCTIONS ON COMPACT SETS

- MAX/MIN PRINCIPLE FOR HARMONIC FUNCTIONS ON COMPACT SETS

- MEAN VALUE PROPERTY FOR HOLOMORPHIC AND HARMONIC FUNCTIONS
  \[ f(z_0) = \frac{1}{2\pi i} \int_{C_{z_0}} f(z) \, dz \]
  \[ u(x, y) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z(\alpha)) \, d\alpha \]
- **Dirichlet's Problem**
  \[ U(r^2, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{U(r^2, \theta')}{r^2 + r'^2 - 2rr' \cos(t-\theta')} dt \]

4. **Series and Residues**

- **Convergence**: Point-wise, uniform, uniform on compact sets (LU)
- **Existence of Taylor Series of \( f \) at \( z_0 \)**
  - Of radius \( r_0 \) when \( f \) is holomorphic on \( D_r(z_0) \)
  - LU convergence of series
  - Should know geometric series, \( e^z, \sin(z), \cos(z) \)
- **Laurent Series Expansion** → **Annulus of Convergence**
- **Classification of Isolated Singularities**
  - Removable
  - Pole
  - Essential → **Picard's Thm** → What happens to \( f(z) \) as \( z \to z_0 \)?
- **Residues and Cauchy Residue Theorem**
  → **Residue Lemmas for Computing Residues**

- **Computing Real Integrals**
  → **Trig Integrals** \( \int_0^\pi P(\cos \theta, \sin \theta) d\theta \)
  → **Cauchy Principal Value Integrals**
  \[ \text{P.V.} \int_{-\infty}^{\infty} f(x) \, dx \quad \text{when} \quad f(x) = \frac{p(x)}{Q(x)} \leq \frac{C}{|x|^2} \quad \text{as} \quad |x| \to \infty \]
  → **Using "Big O" Method and "Big D" Method with Shift Hops**
  → **Breaking Up \( \sin(x) \), \( \cos(x) \) into \( e^{i\pi x} \) to use upper or lower half plane.**
BRING YOUR ID TO FINAL EXAM

LAST OFFICE HOURS:
10:30-12, 1-4:30
M, T, W DEC. 9th-11th, JEFFERY 512.

NO CALCULATORS