**Lemma:** If $f$ has a pole of order $k$ at $z_0$, then

\[ m < k \quad \lim_{z \to z_0} \left| (z - z_0)^m f(z) \right| = \infty, \text{ has limit if } m = k \]

\[ m > k \quad \lim_{z \to z_0} \left| (z - z_0)^m f(z) \right| = 0. \]

**Proof:** By previous lemma:

\[ \lim_{z \to z_0} \left| (z - z_0)^m f(z) \right| = \lim_{z \to z_0} \frac{g(z)}{(z - z_0)^{k-m}} \quad \text{where } g(z_0) \neq 0. \]

**Theorem:** If $z_0$ is an essential singularity of $f$, and $D^*_r(z_0)$ is a punctured disk at $z_0$, then $f \notin C^1 \text{ (except maybe)}$

\[ \exists z \in D^*_r(z_0) \quad \text{s.t.} \quad f(z) = 0. \]

**Proof:** Beyond the scope of this class.

**Example**

\[ \forall \varepsilon > 0, \forall w \in C \quad \exists z \in D^*_\varepsilon(0) \quad z \neq 0, \quad \frac{1}{z} = w. \]

**Zeros of $f$:** If $f$ is holomorphic, "at" $z_0$ we say $f$ has a zero of order $k > 0$ if the following equivalent things hold:

a) $f(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \ldots \quad a_k \neq 0$ and $a_{k+1} = 0$

b) $f(z_0) = 0$, $f'(z_0) = 0$, $f^{(2)}(z_0) = 0$, $f^{(k)}(z_0) = 0$

\[ a_0 = a_1 = \ldots = a_{k-1} = 0, \quad a_k \neq 0. \]

C) We can write $f(z) = (z - z_0)^k g(z)$ where $g$ is holomorphic at $z_0$ and $g(z_0) \neq 0$. 
**Proof**: See Lemma 1. 

We can say \( z_0 \) is "an order zero zero" if \( f(z_0) \neq 0 \).

Why? So we can generalize the concept of "zeros" to include poles.

**Def**: We say \( z_0 \) is a zero of order \( K \in \mathbb{Z} \)

if \( f(z) = (z - z_0)^K y(z) \) where \( y \) is holomorphic at \( z_0 \) and \( y(z_0) \neq 0 \).

**Residues**: If \( f \) has an isolated singularity at \( z_0 \), its residue is the \( a_{-1} \) coefficient. We say \( \text{Res}(f, z_0) = a_{-1} \).

**Lemma 3**: If \( z_0 \) is an isolated singularity of \( f \) which is holomorphic on \( D_\epsilon^+(z_0) \) then for \( \gamma \) a small circle around \( z_0 \) in \( D_\epsilon^-(z_0) \) then \( \int_{\gamma} f(w) dw = 2\pi i \text{Res}(f, z_0) \).

**Proof**: \( \int_{\gamma} f(w) dw = \sum \int_{\gamma} a_n (w-z_0)^n dw = \sum a_n \int_{\gamma} (w-z_0)^n dw = 2\pi i a_{-1} = 2\pi i \text{Res}(f, z_0) \).

**Def**: \( f \) is meromorphic on \( D \) if \( a_z \neq 0 \) or \( f \) has a pole at \( z_0 \).

**Thm**: **Cauchy's Residue Theorem**: If \( f \) is meromorphic on \( D \) and \( \gamma \) is a simple closed curve in \( D \) with interior in \( D \) and finitely many poles \( \{z_2, z_3, \ldots, z_m\} \) in interior of \( \gamma \) then \( \int_{\gamma} f(w) dw = \sum_{j=1}^{m} 2\pi i \text{Res}(f, z_j) \).
PROOF: BY CAUCHY'S THM!

\[
\begin{align*}
\text{SAME} & \quad \xrightarrow{\text{INTEGRAL AS}} \quad \text{C}_{z_1} \quad \theta_{z_1} \\
\end{align*}
\]

USE LEMMA 3.

**RESIDUE LEMMA:** ASSUME ALL FUNCTIONS ARE HOLONOMIC AROUND \(z_0\)

1. \(f(z) = \frac{g(z)}{h(z)}\) where \(g\) and \(h\) have zero of same order at \(z_0\), then \(\text{Res}(f; z_0) = 0\)

2. \(\lim_{z \to z_0} (z - z_0) f(z)\) exists, \(f(z)\) has at most a simple pole at \(z_0\) and \(\lim_{z \to z_0} (z - z_0) f(z) = \text{Res}(f; z_0)\)

3. \(f(z) = \frac{g(z)}{h(z)}\), \(g(z_0) \neq 0\), \(h(z_0) = 0\), and \(h'(z_0) \neq 0\)

   THEN \(\text{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}\)

4. \(f(z) = \frac{g(z)}{(z - z_0)^2}\), \(g(z_0) \neq 0\) THEN \(\text{Res}(f; z_0) = g'(z_0)\)

5. \(f(z) = \frac{g(z)}{h(z)}\) \(g\) has a zero of order \(l\) at \(z_0\)

   \(h\) has a zero of order \(k\) at \(z_0\)

   THEN \(f\) has a zero of order \(k\) at \(z_0\) and if \(k > 0\)

   \(\text{Res}(f; z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} (z - z_0)^{k-1} g'(z)\) where \(g'(z) = (z - z_0)^k f(z)\).
Proof: Follows from Laurent expansion

1. \( f(z) = \frac{g(z)}{h(z)} = \frac{a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \ldots}{b_k(z-z_0)^k + b_{k+1}(z-z_0)^{k+1} + \ldots} \quad \text{zeros of the same order} \quad a_k, b_k \neq 0 \quad k \in \mathbb{Z} \)

Then \( f(z) = \frac{a_k + a_{k+1}(z-z_0)^1 + \ldots}{b_k + b_{k+1}(z-z_0)^1 + \ldots} \)

so we can let \( f(z_0) = \frac{a_k}{b_k} \)

2. If \( \lim_{z \to z_0} (z-z_0)f(z) = L \)

Then \( \lim_{z \to z_0} (z-z_0)^n a_n/(z-z_0)^n = L \)

so \( \lim_{z \to z_0} a_n(z-z_0)^{n+1} = L \)

\( n+1 \geq 0 \) or limit won't exist

\( a_n = 0 \) for \( n < -1 \)

so \( f \) has, at most, a simple pole, and \( a_{-1} = L \).

3. \( f(z) = \frac{g(z)}{h(z)} = \frac{g(z)}{h(z_0)} = 0, \quad h(z_0) = 0, \quad h'(z_0) \neq 0 \)

\( = \frac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots}{b_1(z-z_0) + b_2(z-z_0)^2 + \ldots} \quad \Rightarrow \quad a_0 \neq 0, \quad b_1 \neq 0 \)

so \( f(z) = \frac{1}{(z-z_0)} \frac{a_0 + a_1(z-z_0) + \ldots}{b_1 + b_2(z-z_0) + \ldots} = \frac{s(z)}{z-z_0} \)

Call this \( s(z) \), holomorphic at \( z_0 \) and \( s(z_0) = 0 \)

so \( \lim_{z \to z_0} (z-z_0)f(z) = \lim_{z \to z_0} s(z) = s(z_0) = \frac{a_0}{b_1} = \frac{g(z_0)}{h'(z_0)} = \text{res}(f, z_0) \)
\( f(z) = \frac{g(z)}{(z-z_0)^2} \quad g(z_0) \neq 0 \) THEN

\[
 f(z) = \frac{g_0 + g_1(z-z_0) + g_2(z-z_0)^2 + \cdots}{(z-z_0)^2} = \frac{g_0}{(z-z_0)^2} + \frac{g_1}{z-z_0} + g_2 + \cdots
\]

so \text{ DEGREE TWO POLE AND } \text{Res} \left( f; z_0 \right) = g_1 = g'(z_0).

\( \Box \) EXPLORED IN ASSIGNMENT II.

\textbf{EXAMPLES:}

a) \text{FIND RESIDUES OF } f(z) = \frac{2}{z^2+1} \text{ AT } z = \pm i

\[
 f(z) = \frac{2}{(z+i)(z-i)} \Rightarrow \text{Holo at } z = i, \text{ NOT ZERO}
\]

\[
 \text{Res} \left( f; z = i \right) = \frac{g(i)}{h'(i)} = \frac{\frac{2}{(i+i)}}{1} = \frac{1}{2} \cdot i
\]

\text{BY RESIDUE LEMMA } \Box \text{, } \text{Res} \left( f; z = i \right) = \frac{2}{(z+i)} = g(z) \Rightarrow \text{Holo at } z = -i, \text{ NOT ZERO}

\[
 f(z) = \frac{2}{(z+i)(z-i)} = \frac{2}{z+i} = \frac{1}{z} \Rightarrow \text{Holo at } z = -i
\]

\[
 \text{Res} \left( f; z = -i \right) = \frac{g(-i)}{h'(-i)} = \frac{1}{1} = i
\]

\text{CAN ALSO USE } \Box \text{, } \text{FIND } \lim_{z \to \pm i} \frac{1}{z} f(z)

b) \text{f(z) = } \frac{\cos(z)}{\sin(z)} \text{, POLES AT } z = n\pi, n \in \mathbb{Z}
Use (3) again, \( \cos \left( \frac{\pi}{n} \right) = \frac{\cos(\pi n)}{\cos(n \pi)} = 1 \).

\[
\frac{d}{dz} \sin(z)
\]

\( f(z) = \frac{z^2}{(z-1)(z+1)} \)

\( \cos(\frac{\pi}{2}, -1) \) is easy, use (2) \( \lim_{z \to 1} \frac{z^2}{z-1} f'(z) = \lim_{z \to 1} \frac{z^2}{z-1} = \frac{(-1)^2}{(-1-1)^2} = -\frac{1}{8} \)

What about \( \cos(\frac{\pi}{2}, 1) \)?

Use (5). Let \( \phi(z) = (z-1)^3 f(z) = \frac{z^2}{z+1} \)

\[
\phi'(z) = \frac{(z+1)2z - z^2}{(z+1)^2} = \frac{z^2 + 2z}{(z+1)^2}
\]

\[
= | - (z+1)^{-2} \]

\[
\phi''(z) = \frac{2}{(z+1)^3}
\]

\[
\lim_{z \to 1} \frac{1}{2!} \phi''(z) = \frac{1}{2} \frac{2}{(1+1)^3} = \frac{1}{8}
\]

\( d) \ f(z) = \frac{4z}{z^4 - 6z^2 + 1} \) roots, first find those of \( x^2 - 6x + 1 \) \( x = \frac{6 \pm \sqrt{32}}{2} \)

\( x = 3 \pm 2\sqrt{2} \)

So for \( z^4 - 6z^2 + 1 \) \( w/e \) have \( z = \pm \sqrt{3 \pm 2\sqrt{2}} \) \( \leq \) four roots, all real
For each root, \( z_0 \), we use (3)

\[ \text{Res}(f, z_0) = \frac{y_{z_0}}{y_{z_0}^{-1} \cdot z_0} = \frac{1}{z_0 - 3} \]

so

\[ \text{Res}(f, \pm \sqrt{3} + 2\sqrt{2}i) = \frac{1}{2\sqrt{2}} \]

\[ \text{Res}(f, \pm \sqrt{3} - 2\sqrt{2}i) = -\frac{1}{2\sqrt{2}} \]

(2) \( f(z) = \frac{e^z}{(z-3)} \) find \( \text{Res}(f, 3) \)

Use (5), \( \psi(z) = (z-3)^4 f(z) = e^z \)

\( \psi(z) = e^z \) so \( \text{Res}(4, 3) = \frac{1}{z-3} \frac{1}{3!} \psi'(3) = \frac{e^3}{6} \)

Can also use Taylor expansion of \( e^z \) at \( z = 3 \)

\[ e^z = \sum_{n=0}^{\infty} \frac{e^3}{n!} (z-3)^n \]

So \( \frac{e^z}{(z-3)^4} = \sum_{n=0}^{\infty} \frac{e^3}{n!} (z-3)^{n-4} \)

So we want the \( n=3 \) coefficient.

Evaluating Integrals: 1216 Integrals

Suppose \( p(x, y) \) is some ratio of polynomials in \( x \) and \( y \) (just variables). So

\[ \text{Ex: } p(x, y) = \frac{1}{1+y^2} \text{ or } p(x, y) = \frac{1}{1+a^2-2ax} \text{, } a \neq \pm 1, \text{ REAL} \]

We want to evaluate

\[ \int_0^{2\pi} p(\cos \theta, \sin \theta) \, d\theta \]

\[ = \int_0^{\pi} \frac{1}{1+\sin^2 \theta} \, d\theta \text{ or } \int_0^{\pi} \frac{1}{1+a^2-2a \cos \theta} \, d\theta \]
