But if \( K C D_{1}(0) \) is compact then it is contained in some \( D_{\delta}(1) \) where \( \delta < 1 \) so we have
\[
|z^n - 0| < |z|^n \leq \delta^n \to 0 \text{ as } n \to \infty.
\]

**Definition:** A power series centered at \( z_0 \)

is an expression of the form:
\[
\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad a_n \in \mathbb{C}
\]

**Theorem (Radius of Convergence):** Given a power series \( \sum a_n (z - z_0)^n \) as above, if \( |z - z_0| < R \) series converges. If \( |z - z_0| > R \) series diverges.

Furthermore, convergence is locally uniform on \( D_R(z_0) \).

**Proof:** Same proofs as real analysis, the modulus just means more now. See section 5.9 of book if curious.

**Theorem:** Taylor's Theorem for holomorphic functions

Let \( f : D \to \mathbb{C} \) be holomorphic and \( D_R(z_0) \subset D \), then the series
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n = f(z) \text{ on } D_R(z_0).
\]
Proof: Fix \( z \in D_f(z_0) \), and \( \delta > 0 \) s.t.

\[ |z - z_0| < \delta < \gamma. \]

Let \( \gamma = z_0 + \delta e^{i\theta} \in T(0, \delta) \) circle of radius \( \delta \) around \( z_0 \).

By CIF:

\[ f(z) = \frac{1}{2\pi i} \int_{C_{\delta}} \frac{f(w)}{w - z} \, dw. \]

But

\[ \frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \frac{w - z_0}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}}. \]

\[ \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{w - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}. \]

Locally uniformly convergent for \( z \in D_f(z_0) \) since

\[ |z - z_0| < |w - z_0| = \delta. \]

So

\[ \frac{1}{2\pi i} \int_{C_{\delta}} \frac{f(w)}{w - z} \, dw = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{C_{\delta}} \frac{f(w)}{(w - z_0)^{n+1}} \, dw = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!}. \]

This is what the term "analytic" means, that a function has a Taylor series expansion.

For complex functions this is equivalent to holomorphic (complexly differentiable on open sets).
WE ALSO HAVE THE CONVERSE COROLLARIES OF WEIERSTRASS'S THM OF HOLONOMIC CONVERGENCE

**COR 1.** **ALL POWER SERIES ARE HOLONOMIC INSIDE THEIR RADIUS OF CONVERGENCE.**

**Proof:** **UNIFORM CONVERGENCE OF HOLONOMIC POLYNOMIALS**

**COR 2.** **DERIVATIVE OF A POWER SERIES IS GIVEN BY TERM-BY-TERM DIFFERENTIATION:**
\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \rightarrow f'(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-1} \]

**Proof:** If \( f_n \rightarrow f \) **UNIFORMLY**, \( f_n^{(k)} \rightarrow f^{(k)} \) **UNIFORMLY**

**COR 3.** **IF ** \( f \) **IS DIFFERENTIABLE ON THE REAL LINE AND IF ** \( f \) **HAS A POWER SERIES EXPANSION, THEN ** \( f \) **IS HOLONOMIC IN THE RADIUS OF CONVERGENCE**

**Proof:** \( f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n \)

**RADIUS OF CONVERGENCE IS UNCHANGED.**

**COR 4.'** **TAYLOR SERIES IS UNIQUE**

**Proof:** \( a_n = \frac{f^{(n)}(z_0)}{n!} \)

A CONSEQUENCE OF **COR 1** **IS THAT RADIUS OF CONVERGENCE IS PRESERVED BY INTEGRATION AND DIFFERENTIATION.**
Example: \( \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \)

Real
Is the Taylor series expansion around \( x = 0 \) of radius 1.

Question
Why radius 1?

Answer 1: Do the ratio or root tests, get 1 as an answer. MEH.

Answer 2: Think of \( \frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2} = f(x) \)

If we extend this to a complex function 
\( f(z) = \frac{1}{1 + z^2} \) where are its poles? \( z = \pm i \)

So we can't make a circle of radius larger than 1 where \( f(z) \) is holomorphic. The same is true for \( \arctan(z) \).

Meanwhile, since \( e^z \), \( \sin(z) \) and \( \cos(z) \) are entire their radii of convergence are infinite.
WE CAN GENERALIZE THE TAYLOR SERIES TO FUNCTIONS THAT HAVE POLES, THIS WILL BE USEFUL WHEN WE GET TO THE RESIDUE THEOREM.

**Theorem (Laurent Series)**

Suppose that \( f \) is holomorphic on the annulus \( \mathbb{D}(\mathbb{C}, D) = \{ z \in \mathbb{C} \mid r < |z - z_0| < R \} \) for some \( r < R \) (\( R \) can be \( \infty \)) THEN

We can write

\[
 f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n
\]

The sum converges on the annulus

\[
 c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{n+1}} \, dw
\]

\( n \) can be negative

\( \gamma \) is a ccw circle with radius between \( r \) and \( R \).

**Note:** Unlike for Taylor series, \( a_n = \frac{f^{(n)}(z_0)}{n!} \)

\( f \) isn't defined at \( z_0 \), but the integrals are!

**Proof:** Let \( \gamma_1 \) and \( \gamma_2 \) be circles of radii \( \rho_1 \) and \( \rho_2 \) respectively, and \( \gamma \subset \rho_1, \rho_2 < R \). Then