\[ f_n = \text{TRIANGLE} \quad \rightarrow \quad f_n(x) \]

so \( \lim_{n \to \infty} f_n = 0 \) but \( \lim_{n \to \infty} \int f_n(x) \, dx = \lim_{n \to \infty} 1 = 1 \)

so pointwise convergence doesn't always preserve continuity or integrals.

Uniform convergence does.

\text{Definition:} \quad f_n: S \to \mathbb{C} (or \mathbb{R}) \text{ converges uniformly on } S \text{ if } \forall \varepsilon > 0 \quad \exists N \quad \text{s.t.}

\[ |f_n(z) - f(z)| < \varepsilon \quad \forall z \in S \]

What's different? This occurs at the end, it doesn't matter what point we use.

We say \( f_n \to f \) uniformly on compact sets in \( S \) if for any compact subset \( K \subset S \), \( f_n \to f \) uniformly on \( K \). Also called locally uniformly convergent (LU).

Often, uniformly on compact subsets is enough to get what we need.

\text{Lemma:} \quad \text{If } f_n \to f \text{ uniformly on compact sets in } S \text{ and } K \subset S \text{, then}

\[ \lim_{n \to \infty} \int_0^2 f_n(z) \, dz = \int_0^2 f(z) \, dz \]
\[
\text{(10)} \quad \left| \int_0^1 f_n(z) \, dz - \int_0^1 f(z) \, dz \right| = \left| \int_0^1 (f_n(z) - f(z)) \, dz \right| \\
\leq \text{length}(\gamma) \cdot \max_{z \in \gamma} |f_n(z) - f(z)|
\]

compact on \( \forall \varepsilon > 0, \exists N \text{ s.t. } |f_n(z) - f(z)| < \varepsilon \text{ on } \gamma \text{ for } n \geq N. \) 

So \( \lim_{n \to \infty} \left| \int_0^1 f_n(z) \, dz - \int_0^1 f(z) \, dz \right| \leq \text{length}(\gamma) \cdot \varepsilon < \varepsilon = 0. \)

**Cor.** If \( f(z) = \sum_{k=1}^{\infty} g_n(z) \to f(z) = \sum_{k=1}^{\infty} g(z) \) \text{ uniformly on compact sets on } \mathbb{C}, \text{ then } \forall \varepsilon > 0, \exists N \text{ s.t. } |g_n(z) - g(z)| < \varepsilon \text{ on } \mathbb{C} \text{ for } n \geq N.

\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} g_n(z) = \lim_{n \to \infty} \int_0^1 f_n(z) \, dz = \sum_{k=1}^{\infty} \int_0^1 f(z) \, dz = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} g_n(z)
\]

so \( \sum_{n=1}^{\infty} g_n(z) \, dz = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} g_n(z) \, dz. \)

**Theorem:** (Due to Weierstrass) If \( f_n \to f \) \text{ uniformly on } \text{compact } D \text{ and } f_n \text{ are harmonic on } D \text{ then so is } f \text{ and } f_n \to f \text{ on } D.

**Proof:** Fix \( z_0 \in D, D(z_0) \text{ a small disk around } z_0 \text{ contained in } D. \)

By Cauchy's Integral Formula:
\[
F_n(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w)}{w-z_0} \, dw 
\]

let \( \gamma = \partial D(z_0). \)
\[
\frac{f(z_0)}{w-z_0} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z_0} \, dw
\]

Since if \( f_n(w) \to f(w) \) uniformly on \( D(z_0) \), then
\[
\frac{f_n(w)}{w-z_0} \to \frac{f(w)}{w-z_0} \quad \text{uniformly on} \quad \gamma.
\]

So for all \( z_0 \in D \), \( f(z) \) is given by an integral of Cauchy type, thus analytic.

Since
\[
\frac{k_1}{2\pi i} \int_{\partial} \frac{f_n(z)}{\gamma(w-z)^{n+1}} \, dz \to \frac{k_1}{2\pi i} \int_{\gamma} \frac{f(z)}{(w-z)^{n+1}}
\]

It is easy to show uniform convergence for the derivative.

**Distinction between uniformly convergent and uniformly convergent on compact sets**

For \( z \in D(0) \), \( f_n(z) = z^n \to 0 \) uniformly on compact sets, but not uniformly

because \( \forall \varepsilon > 0 \) we can find a \( z \in D(0) \)

\[|z^n - 0| > \varepsilon.\] Just look near the boundary.
but if \( k \in \mathbb{D}(0) \) is compact then it is contained in some \( \overline{D}(1) \) where \( \delta < 1 \) so we know
\[ |z^n - 0| < |z|^n \leq \delta^n \rightarrow 0 \text{ as } n \rightarrow \infty. \]

**Definition:** A **power series** centered at \( z_0 \) is an expression of the form:
\[ \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad a_n \in \mathbb{C} \]

**Theorem (Radius of Convergence):** Given a power series \( \sum_{n=1}^{\infty} a_n(z-z_0)^n \) (as above)
series converges. If \( |z-z_0| > \rho \), series diverges.

Furthermore, convergence is locally uniform on \( D_\rho(z_0) \).

**Proof:** Same proofs as real analysis, the modulus just means more now. See section 5.4 of book if curious.

**Theorem:** Taylor's Theorem for Holomorphic Functions

Let \( f : D \rightarrow \mathbb{C} \) be holomorphic and \( D_r(z_0) \subset D \), then the series \( \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z) \) on \( D_r(z_0) \).