\[ f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + ce^{it}) \, dt \quad \text{where} \quad z = x + iy. \quad (9) \]

So
\[ u(x_0, y_0) + iv(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos t, y_0 + r\sin t) \, dt \]
\[ + i v(x_0 + r\cos t, y_0 + r\sin t) \, dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos t, y_0 + r\sin t) \, dt \]
\[ + i \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + r\cos t, y_0 + r\sin t) \, dt \]

Comparing real and imaginary parts, we are done.

**Theorem**: (Local Max/Min Principle for Harmonic Functions)

**Note**: There's no modulus for harmonic functions, they are real-valued.

If \( u : \Omega \to \mathbb{R} \) is harmonic and \( \exists \) a local max/min at \( (x_0, y_0) \) in \( \Omega \) then \( u \) is constant on an open disk around \( (x_0, y_0) \).

\[ u(x_0, y_0) \geq u(x, y) \quad \forall (x, y) \in D_\delta(x_0, y_0) \]

or \( u(x_0, y_0) \leq u(x, y) \quad \forall (x, y) \in D_\delta(x_0, y_0) \)

**Proof**: For local max, same as for holomorphic functions, just need MVP and continuity. For local min, use local max argument on \(-u\).
THEOREM: (Local Max/Min Principle for Harmonic Functions)

$u : D \to \mathbb{R}$ is harmonic on $D$, $cts$ on $\overline{D}$

Let $M = \max_{(x,y) \in \partial D} u(x,y)$, $m = \min_{(x,y) \in \partial D} u(x,y)$ then

$a) \quad m \leq u(x_0,y_0) \leq M \quad \forall (x_0,y_0) \in \partial D$

$b) \quad \text{If } u(x,y) = M \text{ or } m \text{ for some } (x,y) \in D \text{ then } u \text{ is constant on } D.$

Proof: Same as for Max Modulus, just use local max/min Principle instead of local max mod.

Dirichlet Problem

In applications of harmonic functions (see physics), we are faced with the problem of, given a bounded domain $D$ and a $cts$ function $u : \overline{D} \to \mathbb{R}$, extending $u$ to a harmonic function on $D$ which is $cts$ on $\overline{D}$. 
For example, given the temperature on the boundary of a slab, we want to find the temp. on the slab.

This problem is the Dirichlet problem.

**Lemma:** Solutions to Dirichlet problem are unique if they exist.

**Proof:** Suppose \( u, \tilde{u} : D \to \mathbb{R} \) are cts, agree on \( \partial D \) and are harmonic on \( D \). Then \( u - \tilde{u} = 0 \) everywhere on \( \partial D \) and by global max/min principle \( 0 \leq u - \tilde{u} \leq 0 \) on \( D \). So \( u = \tilde{u} \).

If \( D \) is a disk, Dirichlet problem is solved via an explicit formulation known as Poisson's formula.

The idea is that if we knew \( u \) we could find a harmonic conjugate, \( v \), so that \( f = u + \overline{v} \) and then \( u(x, y) = \mathbb{R}^2 \int_{\partial D} \overline{v(w)} \frac{1}{|w - (x, y)|} \, d\nu \).
We want the right hand side to only depend on \( u \), not \( v \).

**Theorem**: (Poisson's Formula) If \( u \) is harmonic on a domain containing the closed disk \( D_R(0) \) for \( R > 0 \), then \( u(x + iy) \) for \( D_R(0) \):

\[
U(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{it}) \frac{R^2 - r^2}{|Re^{it} - z|^2} \, dt
\]

Letting \( z = Re^{ix} \) in polar coordinates this is

\[
U(Re^{ix}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u_0(Re^{it})}{R^2 - 2Rr\cos(t - x) + r^2} \, dt
\]

**Proof**: Let \( v \) be the harmonic conjugate for \( u_0 \) on a simply-connected set containing \( D_r(0) \). So \( f = u + iv \)

\[
f(z) = \frac{1}{2\pi} \int \frac{f(w)}{w - z} \, dw
\]

where \( C \) is the counterclockwise circle of radius \( R \).

Now consider \( f(w) = \frac{f(w)z}{w - z} \) which is holomorphic in \( w \) on \( D_{1}(0) \) (for fixed \( z \) with \( |z| < R \)) where \( R_1 > R \), slightly greater.

Since \( R^2 - |z|^2 \neq 0 \) on \( D_1(0) \) or on its boundary,

\[
\frac{1}{2\pi} \int \frac{f(z)}{w - z} \, dw = 0
\]

by Cauchy's THM.

So

\[
f(z) = \frac{1}{2\pi i} \int \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi i} \int \frac{f(w)(R^2 - |z|^2)}{(w - z)(R^2 - R_1^2)} \, dw
\]

\[
= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(Re^{it})(R^2 - |z|^2)}{(Re^{it} - z)(R - Re^{it}z)} \, R ye^{it} \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{it})(R^2 - |z|^2)}{(R - ze^{it}) (R - Re^{it}z)} \, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{it})(R^2 - |z|^2)}{1 - Rze^{-it}|^2} \, dt
\]