**Theorem 1 (Cauchy-Type Integrals are Analytic)**

If \( \gamma \) is a curve (not necessarily a loop) and \( g : \gamma \to \mathbb{C} \) is continuous, then

\[
G(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)}{w-z} \, dw
\]

is analytic on \( \mathbb{C} \setminus \gamma \) and

the \( k \)-th derivative is given by

\[
G^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{g(w)}{(w-z)^{k+1}} \, dw
\]

**Heuristic Idea:** See homework problem 1.

**Proof:** Later.

**Theorem:** (Cauchy's Differentiation Formula) (CDF)

If \( f : D \to \mathbb{C} \) is analytic on simply connected \( D \) then \( f \) is infinitely differentiable on \( D \) and

for each \( \gamma \in D \)

\[
f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{k+1}} \, dw
\]

for any closed loop \( \gamma \subset D \)

with \( n(\gamma, z) = 1 \).

**Proof:** Follows from part 2 and analyticity of Cauchy type integrals.

\[
\text{Ex. } \sum_{n=1}^{\infty} \frac{e^{z}}{z^n} \, dz = 2\pi i \left( \frac{z}{2\pi} \int_{1}^{z} \frac{e^{\zeta}}{(\zeta - z)^{n+1}} \, d\zeta \right) = 2\pi i \int_{1}^{z} \frac{d}{d\zeta} \left( \frac{e^{\zeta}}{\zeta} \right) \, d\zeta = 2\pi i \quad \text{for } z > 1
\]
Proof of Theorem 1 (for first derivative only)

\[ G(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)}{w-z} \, dw, \quad z \in \mathbb{C} \setminus \gamma \]

\[
\frac{G(z+h)-G(z)}{h} = \frac{1}{h} \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)}{w-(z+h)} - \frac{g(w)}{w-z} \, dw
\]

\[
= \frac{1}{h} \frac{1}{2\pi i} \int_{\gamma} \frac{h g(w)}{(w-(z+h))(w-z)} \, dw = \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)}{(w-(z+h))(w-z)} \, dw
\]

Want to show this converges to \( \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)}{(w-z)^2} \, dw \) as \( h \to 0 \)

Well \( |(w-(z+h))(w-z)^2| \geq \delta^3 \) so

\[
\frac{1}{h} \int_{\gamma} \frac{g(w)}{(w-(z+h))(w-z)^2} \, dw - \frac{1}{h} \int_{\gamma} \frac{f(w)}{(w-(z+h))(w-z)^2} \, dw = \frac{1}{2\pi i} \int_{\gamma} \frac{h g(w)}{(w-(z+h))(w-z)^2} \, dw
\]

Uniform convergence argument

Let \( L = \text{length}(\gamma) \) and \( M = \max_{w \in \gamma} |g(w)| \), then

\[
\left| \frac{1}{h} \int_{\gamma} \frac{h g(w)}{(w-(z+h))(w-z)^2} \, dw \right| \leq \frac{L}{2\pi} \frac{M}{\delta^3} \xrightarrow{h \to 0} 0. \text{ Giving results.}
\]

Argument for higher derivatives follows similarly from induction.
\textbf{Cor. 1:} A holomorphic function is infinitely differentiable.

\textbf{Proof:} If \( f : D \to \mathbb{C} \) is holomorphic on some open set, then for any \( z_0 \in D \), \( \exists \rho > 0 \) s.t. \( D_\rho(z_0) \subset D \) and disks are simply connected. Use CDF.

\textbf{Cor. 2:} Cauchy's Inequality for Derivative

If \( f : D \to \mathbb{C} \) is holomorphic and a circle \( \gamma \subset D \) has radius \( R \) centered at \( z_0 \) and \( \max_{\gamma} |f'(z)| \leq M \), then

\[
|f^{(k)}(z_0)| \leq \frac{k! M}{R^k}.
\]

\textbf{Proof:} \( |f^{(k)}(z_0)| = \left| \frac{k!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{k+1}} dw \right| \leq \frac{k!}{2\pi} \cdot \frac{M}{R^{k+1}} \cdot 2\pi R = \frac{k! M}{R^k}. \)

\textbf{Cor. 3:} Liouville's Theorem

The only bounded entire functions are constant.

\textbf{Proof:} If \( f : \mathbb{C} \to \mathbb{C} \) is holomorphic and bounded \( \exists M > 0 \) s.t. \( |f(z)| \leq M \) for all \( z \in \mathbb{C} \). So by \textbf{Cor. 2:} \( |f'(z)| \leq \frac{M}{R} \) for all \( R > 0 \), \( \forall z \in \mathbb{C} \).

Since we can take \( R \) arbitrarily large.
So \( f(z) = 0 \) for all \( z \in \mathbb{C} \), so \( f(z) \) is constant.

**Cor. 4. The Fund. THM. of Algebra (FTA):**

Every polynomial on \( \mathbb{C} \) factors completely into linear components.

**Proof:** If \( p(z) = a_nz^n + \cdots + a_1z + a_0 \) a poly. of degree \( n \geq 1 \)
then if \( p(\alpha) = 0 \), \( p(z) - p(\alpha) = q(z)(z - \alpha) \)
where \( q(z) \) is a poly. of deg. \( n - 1 \).

Why? Polynomial long division.

So it is enough to show \( p(\alpha) = 0 \) for some \( \alpha \in \mathbb{C} \), then \( p(z) = q(z)(z - \alpha) \) and we have reduced the problem to degree \( n - 1 \) and we can proceed by induction until \( q(z) \) is degree 1 (linear).

So FTA \( \implies p(\alpha) = 0 \) for \( \alpha \in \mathbb{C} \).

**Proof by Contradiction:**
Suppose \( p(\alpha) = 0 \) for all \( \alpha \in \mathbb{C} \). Then \( f(z) = \frac{1}{p(z)} \) is entire.

Since \( |p(z)| = |a_n||z^n| - |a_{n-1}| |z|^{n-1} - \cdots - |a_1||z| + |a_0| \)
we have \( |p(z)| \geq \frac{|a_n||z|^n}{2} \) for sufficiently large \( |z| \geq R > 0 \)
so \( |f(z)| = \frac{1}{|p(z)|} \geq \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n|^n} \) for \( |z| \geq R \).

And \( M = \max_{|z| \leq R} |f(z)| \) exists since \( f(z) \) is CTS everywhere and \( |z| \leq R \) is compact.

So \( |f(z)| \leq \max(M, \frac{2}{|a_n|^n}) \), so \( f(z) \) is bounded thus constant.