If \( f : \mathbb{D} \to \mathbb{C} \) is holomorphic

\[ \begin{array}{c}
\text{Points not in } D \\
\end{array} \]

SAME INTEGRAL

\[ \begin{array}{c}
\text{Both directions cancel} \\
\end{array} \]

So all integrals on simple closed curves can be considered as on circles around bad points.

Proof of version 3:

\[ \begin{array}{c}
\text{Go to } \gamma_1 \\
\end{array} \]

\[ \begin{array}{c}
\text{Go to } -\gamma_0 \\
\end{array} \]

We want to show \( \int + \int + \int - \int = 0 \)

\[ \begin{array}{c}
\int_{\gamma_1} + \int_{\gamma_1} - \int_{\gamma_0} = 0 \\
\end{array} \]

So \( \int_{\gamma_1} = \int_{\gamma_0} \).
In reality though, unlike in version 2, the picture isn't that nice.

Something like this is perfectly allowable.

So what do we do?

Well \([0,1] \times [0,1]\) is compact in \(\mathbb{R}^2\)

(a closed, bounded set in \(\mathbb{R}^2\))

which means

\[ H: [0,1] \times [0,1] \rightarrow \mathbb{D} \text{ is uniformly continuous} \]

(continuous on a compact set gives uniform continuity)

This means that \(\exists \delta > 0\) such that if

\[ ||(s_1,t_1) - (s_2,t_2)|| < \delta \]

then \[ ||H(s_1,t_1) - H(s_2,t_2)|| < \varepsilon \].

This is different from regular continuity since there is one \(\delta > 0\) for every point in \([0,1]\).

Furthermore, since \([0,1] \times [0,1]\) is compact and \(H\) is cts, the image of \([0,1] \times [0,1]\) under \(H\) is compact (closed and bounded in \(\mathbb{C}\)) so

there exists a minimum distance of \(H([0,1] \times [0,1])\) from \(\mathbb{C} \setminus \mathbb{D}\), call this \(\varepsilon > 0\).
Now chop \([0, 1] \times [0, 1]\) into squares with diagonal length \(2\delta\).

Taking all of these paths together, they add up to \(-\delta_0 + \delta + \gamma_1 - \gamma = \sum \gamma_i\).

But each square in \([0, 1] \times [0, 1]\) is contained in a disk of radius \(\delta\).

And so maps to a disk of radius \(\varepsilon\) which is contained in \(D\). But this is a closed loop in a disk where \(f(z)\) is holomorphic, by the Cauchy integral \(\int f(z)dz = 0\) for all \(\gamma_i\). So \(\sum S + S + S + S = -S + S = 0\) so \(\int S = 0\).
**Corollary 3:** If $D$ is a simply connected domain and $f: D \to \mathbb{C}$ is analytic, then $f$ has an anti-derivative on $D$. That is, $\exists F: D \to \mathbb{C}$ s.t. $F'(z) = f(z)$.

**Proof:** Since $D$ is simply connected, every closed curve in $D$ is homotopic to a point so $\int_{\gamma}(z)\,dz = 0$ by (TV3). Thus by equivalence, $f(z)$ has an anti-derivative.

**Corollary 4:** If $D$ is simply connected and $f: D \to \mathbb{C}$ is holomorphic and never zero on $D$, then $\exists g: D \to \mathbb{C}$ holomorphic such that $f(z) = e^{g(z)}$.

**Proof:** Next week we'll see that $f(z)$ is holomorphic if $f(z)$ is holomorphic, so $g(z) = \frac{f'(z)}{f(z)}$ is holomorphic on $D$ (since $f(z)$ is on $D$). So by (TV2), $\exists G: D \to \mathbb{C}$ s.t. $G'(z) = g(z)$.

So consider

$$h(z) = e^{-G(z)}f(z)$$

$$h'(z) = -G'(z)e^{-G(z)}f(z) + e^{-G(z)}f'(z) = -\left(\frac{f'(z)}{f(z)}\right)e^{-G(z)}f(z) + e^{-G(z)}f'(z)$$

$$= f'(z)e^{-G(z)} - f'(z)e^{-G(z)} = 0$$

So $h(z)$ is constant on $D$ so $h(z) = e^{-G(z)}f(z) = c$.

Let $\zeta$ be such that $e^{\zeta} = c$ then $f(z) = e^{\zeta}e^{G(z)} = \zeta + z$ proving the corollary.

**Corollary 5:** On any simply connected domain $D$ there is a function $g(z)$ so that $e^{g(z)} = z$.

**Proof:** Use Cor 4 on $f(z) = z$. This gives us a branch of log on any simply connected domain that does not contain 0.