In constructing other proofs in the future, it will be useful to bound the integral.

**Definition.** Given \( \gamma : (a, b) \to \mathbb{C} \), with \( \gamma(t) = x(t) + iy(t) \), then

\[
\text{length}(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \int_a^b |\gamma'(t)| \, dt \quad \text{(old arc length definition)}
\]

**Proposition 3:** If \( f \) is a complex function defined on a open neighborhood or a curve \( \gamma \), if \( |f(z)| \leq M \) for all \( z \in \gamma \), then

\[
\left| \int_\gamma f(z) \, dz \right| \leq M \text{length}(\gamma)
\]

**Proof:** Recall

\[
\left| \int_\gamma f(z) \, dz \right| = \lim_{n \to \infty} \left| \sum_{i=1}^{n} \frac{f(\gamma(t_i))(\gamma(t_i) - \gamma(t_{i-1}))}{n} \right|
\]

\[
\leq \lim_{n \to \infty} \sum_{i=1}^{n} |f(\gamma(t_i))(\gamma(t_i) - \gamma(t_{i-1}))| \leq M \lim_{n \to \infty} \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \leq M \text{length}(\gamma)
\]

Any sequence of straight paths will be shorter.

**Example:**

\[
\left| \int_\gamma e^z \, dz \right| = e \pi \text{ for } \gamma(t) = e^{it}, \ t \in [0, \pi].
\]

Since \( |e^z| = e^x \leq e \) on top-half circle and length(\(\gamma\)) = \(\pi\).
Remark: We can do better by splitting up the curve.

\[ \int_0^\pi \cos^2 z \,dz \leq \frac{\pi}{2} + \frac{\pi}{2}. \]

So \( |\int_0^\pi \cos^2 z \,dz| \leq \frac{\pi}{2} + \frac{\pi}{2}. \)

\[ 0 \leq 1 \leq 1 \]

| \[ \begin{align*}
\text{Fundamental Theorem of Complex Calculus} \\
\text{Suppose } \gamma : [a,b] \to \mathbb{C} \text{ is a piecewise } C^1 \text{ curve, and that } f : \mathbb{C} \to \mathbb{C} \text{ is a domain in } \mathbb{C} \\
\text{such that } f'(z) = f(z), \text{ then if } \gamma(a) = z_0 \text{ and } \gamma(b) = z, \text{ then} \\
\int_{\gamma} f(z) \,dz = F(z) - F(z_0). \\
\text{Proof: Let } \gamma(t) = (x(t), y(t)) \\
\int_{\gamma} f(z) \,dz = \int_{a}^{b} f(\gamma(t)) \gamma'(t) \,dt = \int_{a}^{b} f'(y(t)) \gamma'(t) \,dt \\
\text{Now } F(x + iy) = U(x,y) + iV(x,y) \text{ so} \\
\frac{d}{dt} \left( F(\gamma(t)) = \frac{d}{dt} \left( U(x(t),y(t)) + iV(x(t),y(t)) \right) \right) \\
= \frac{\partial U}{\partial x} (x(t)) \frac{dx}{dt} + \frac{\partial V}{\partial x} (x(t)) \frac{dy}{dt} + i \left( \frac{\partial U}{\partial y} (y(t)) \frac{dy}{dt} + \frac{\partial V}{\partial y} (y(t)) \frac{dx}{dt} \right) \]
\[
\frac{dx}{dt} \left( \frac{dx}{dy} + i \frac{dy}{dy} \right) + i \frac{dx}{dt} \left( \frac{2v}{dy} - i \frac{2u}{dy} \right) = 0 \\
\frac{dx}{dt} \left( F' \left( \gamma(t) \right) \right) + i \frac{dx}{dt} \left( F' \left( \gamma(t) \right) \right) = F' \left( \gamma(t) \right) \gamma'(t) \Rightarrow \leq 0 \\
\int_{z_0}^{z_1} F'(\gamma(t)) \, dt = F(\gamma(t)) \bigg|_{t=0}^{t=1} = F(z_1) - F(z_0) \\
\text{by the regular FTC} \\
\]

\underline{Example: Recall} \quad \int_{\gamma} z^2 \, dz \quad \text{where} \quad \gamma(t) = 2t + i(1-t) \quad \text{for} \quad t \in [0,1] \\

We did this before the hard way, but \;
\frac{1}{3} z^3 \quad \text{is an anti-derivative of} \quad z^2 \quad \text{so} \;
\int_{\gamma} z^2 \, dz = \left[ \frac{z^3}{3} \right]_i^i = \frac{8}{3} + i \left( \frac{1}{3} \right) \quad \text{which is the same answer we got last time.} \\

So now one may ask: When does a complex function have an anti-derivative on a domain? \;

Well, we can verify when one \underline{doesn't} have one \underline{by} seeing if an integral isn't \underline{path-independent}. \quad
If \( f \) has an antiderivative \( F \) on \( S \), domain containing \( \gamma \), then \( \int_{\gamma} f(z) \, dz \) only cares about the endpoints.

So if two different paths give different integrals, then \( F \) must not exist everywhere on \( S \).

**Ex:** Let \( S = \mathbb{C} - \{0\} \), \( f(z) = \frac{1}{z} \)

\[ \gamma_1(t) = \cos t + i \sin t \quad t \in [0, \pi] \]
\[ \gamma_2(t) = 10 \cos t - 3 \sin t \quad t \in [0, \pi] \]

Easy to check (using either old way of computing integrals, or new way using \( \frac{d}{dz} \log z = \frac{1}{z} \) branch or \( \log \)) that

\[ \int_{\gamma_1} \frac{1}{z} \, dz = i \pi i - i \pi = \int_{\gamma_2} \frac{1}{z} \, dz \]

We'd like to know if path-independence tells us if a function has an anti-derivative.
Theorem 2: Let \( f: D \to \mathbb{C} \) be continuous on its domain. The following are equivalent:

i) Integrals are path-independent: If \( z_0, z_1 \in D \) and \( \gamma_1, \gamma_2 \subseteq S \) are piecewise \( C^1 \) paths that both join \( z_0 \) to \( z_1 \), then \( \int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz \) give same integrals.

ii) Integrals on closed loops in \( D \) are zero. If \( \gamma: [a, b] \to D \) is \( C^1 \) and \( \gamma(b) = \gamma(a) \) then \( \int_{\gamma} f(z) \, dz = 0 \).

iii) There is an anti-derivative of \( f \) on \( D \). That is, \( \exists F: D \to \mathbb{C} \) such that \( \frac{d}{dz} F(z) = f(z) \) on \( D \).

Proof: (iii) \( \Rightarrow \) (i) by the Fundamental Theorem of Complex Calculus.

(i) \( \Leftrightarrow \) (iii) is easy because every loop can be split into two paths.
\[
\begin{align*}
\oint_{\gamma_1} f(z) \, dz &= \oint_{\gamma_2} f(z) \, dz + \oint_{\gamma_3} f(z) \, dz = \oint_{\gamma_0} f(z) \, dz - \oint_{\gamma_2} f(z) \, dz = 0
\end{align*}
\]

\[
\Rightarrow
\]

\[
\int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz
\]

So we just need to show \((i) \Rightarrow (iii')\).

Suppose \((i)\). Fix \(z_0 \in D\). \(\forall z \in D\) \exists \gamma \subseteq D, \text{not unique, such that } f\text{, flows } z_0 \text{ to } z = \gamma_{z_0}

Define \(F(z) = \int_{\gamma_{z_0}} f(w) \, dw\) since integrals are path-independent this is well-defined.

We want to show \(F'(z) = f(z)\) on \(D\). This will prove our claim.

Fix \(z \in D\), we want to show

\[
\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z), \text{ or equivalently}
\]

\[
\lim_{h \to 0} \left| \frac{F(z+h) - F(z) - h f(z)}{h} \right| = 0
\]
Fix $\varepsilon > 0$, since $F$ is continuous (cts) at $z$, $\exists \delta > 0$ s.t. $|f(y) - f(z)| < \varepsilon$ when $|w - z| < \delta$.

Let $|w| < \delta$.

Let $\gamma_h$ be a straight path from $z$ to $z + h$.

Then since $F(z) = \int_{\partial \gamma} f(w) \, dw$ and $F(z + h) = \int_{\partial (\gamma + h)} f(w) \, dw$,

$F(z + h) - F(z) = \int_{\gamma + h} f(w) \, dw$.

Furthermore, $F(z) \ h = f(z) \int_{\gamma} 1 \, dw = \int_{\gamma} f(z) \, dw$.

So $F(z + h) - F(z) - F(z) \ h = \int_{\gamma} (f(w) - f(z)) \, dw$.

But $|w - z| < \delta$ for $w \in \gamma_h$ so $|f(w) - f(z)| < \varepsilon$.

So $|\int_{\gamma} f(w) - f(z) \, dw| \leq \varepsilon$ (length $\gamma_h = 3|h|$).

So $\left| \frac{\int_{\gamma} f(w) - f(z) \, dw}{h} \right| \leq \frac{\varepsilon |h|}{|h|} = \varepsilon$.

So $\lim_{h \to 0} \left| \frac{F(z + h) - F(z) - f(z) \ h}{h} \right| = 0$, giving the result.

$\square$
Now we will show that we have similar flexibility for integrating if we let $f(z)$ be complex differentiable on a simply connected domain.

**DEF:** A contour $\gamma : [a,b] \rightarrow \mathbb{C}$ is called a **simple closed curve** if $\gamma(a) = \gamma(b)$ and it does not otherwise cross itself.

It seems intuitive that a simple closed curve always has an "interior" and an "exterior".

But it's not mathematically trivial. It's called the "Jordan curve theorem," and we'll just assume it.
A simple closed curve is **positively oriented** if

the interior is to the left of the direction of the curve's parametrization

otherwise **negatively oriented**.

**Cauchy's Theorem (Version 1)**:

Supp. D is a domain and \( f : D \to \mathbb{C} \)

is holomorphic. If \( \gamma \subseteq D \) and its interior is also in \( D \) then

\[
\oint_{\gamma} f(z) \, dz = 0.
\]

**Remark**: Note this theorem doesn't apply to \( f(z) = \frac{1}{z} \) on \( C - \{0\} \) with \( \gamma(t) = e^{it} + t \in [0, 2\pi] \), since \( 0 \in \text{interior}(\gamma) \).
PROOF (probably a bit unsatisfying since it uses vector calc.)

We'll again assume that partials of \( u, v \) are continuous (cheating... see note)

Recall Green's theorem:

\[
\int _R P(x,y)dx + Q(x,y)dy = \iint _R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy
\]

If partials exist and are cts.

Let \( f(x+iy) = u(x,y) + i v(x,y) \) \( \gamma(t) = x(t) + iy(t) \)

Then \( \int \gamma(t) \, dt = \int _0 ^b f(\gamma(t)) \gamma '(t) \, dt \)

\[
= \int _0 ^b \left( u(x(t),y(t)) + iv(x(t),y(t)) \right) (x'(t) + iy'(t)) \, dt
\]

\[
= \int _\partial \left( iu + v \right) + \int _R (2v_x - 2u_y) dxdy
\]

= 0
NOTE: THERE IS A PROOF OF THIS THEOREM (DUE TO GOURSAT) THAT DOES NOT ASSUME CONTINUITY OF PARTIALS, WHICH IS GOOD BECAUSE WE EVENTUALLY USE THE RESULTS OF THIS THEOREM TO PROVE THE PARTIALS ARE INFinitely DIFF-ABLE. THE PROOF IS SUPPRESSED SINCE IT WOULD TAKE DAYS.

COROLLARY 1: IF $f$ IS HOLONOMIC ON THE DISK $D_p(z_0)$, THEN $f$ HAS AN ANTI-DERIVATIVE ON $D_p(z_0)$.

PROOF: THE PROOF OF THE EQUIVALENCE THEOREM (YESTERDAY?) BETWEEN ANTI-DERIVATIVE EXISTANCE AND PATH INDEPENDENCE CAN BE EASILY ADAPTED.

LET $y_2$ BE A STRAIGHT PATH FROM $z_0$ TO $z$ AND LET

$$F(z) = \int_{y_2} f(w) \, dw.$$ FOR $|w| < \varepsilon$, $\varepsilon$ IS SUFFICIENTLY SMALL.