PROPOSITION I: THE FUNCTION $\log z$ IS DIFFERENTIABLE ON $S = \mathbb{C} - (-\infty, 0]$, WITH $\frac{d}{dz} \frac{1}{z} = \frac{i}{z}$.

Proof: $\log(x+iy) = \frac{1}{2} \ln |x+iy|^2 + i \arctan \left( \frac{y}{x} \right)$.

When $x > 0$, $u(x, y) = \ln |x+iy|^2$ and $v(x, y) = \arctan \left( \frac{y}{x} \right)$.

\[
\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = -\frac{y}{x^2+y^2},
\]
\[
\frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}, \quad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}.
\]

So $\log(z)$ IS DIFFERENTIABLE IN THIS REGION AND

\[
\frac{d}{dz} (\log(z)) = \frac{1}{2} \frac{x}{x^2+y^2} + i \frac{y}{x^2+y^2} = \frac{1}{z},
\]

For $y > 0$, $\log(x+iy) = \frac{1}{2} \ln |x+iy|^2 + i \arccot \left( \frac{x}{y} \right)$, AND $\frac{d}{dt} \arccot(t) = -\frac{1}{1+t^2}$

same argument gives $\frac{d}{dz} (\log(z)) = \frac{1}{z}$ IN THIS REGION.

For $y < 0$, $\log(x+iy) = \frac{1}{2} \ln |x+iy|^2 - i \arccot \left( \frac{x}{y} \right)$. SAME ARGUMENT.

These three regions cover $S$.

COROLLARY I: $\log_+(z)$ IS DIFFERENTIABLE ON $S = \mathbb{C} - \mathbb{R}^+$ NOT ON $\cup$ cut? AND $\frac{d}{dz} \log_+(z) = \frac{1}{z}$.

Proof: If $\gamma = -\pi + 2\pi ik$ THEN $\log_+(z) = \log(z) + 2\pi ik$, so it's true.

Otherwise, if $z \in \mathbb{S}$ and $z \notin (-\infty, 0]$ THEN $\exists \delta > 0$ and $k \in \mathbb{Z}$ s.t. $\log_+(z) = \log(z) + 2\pi ik$ on $D_\delta (\mathbb{S})$, so result is true. If $z \in \mathbb{S}$ and $z \notin (-\infty, 0]$ THEN $\exists \delta > 0$ and $\forall k \in \mathbb{Z}$ s.t. $\log_+(z) = \frac{1}{2} \ln |x^2+y^2| + i(\arctan \frac{y}{x} + \pi + 2\pi k)$, Proof follows as in Prop I.
Observe this proof holds even if the branch cut isn't a line.

**Corollary 2:** Suppose that $S$ is an open set and that $\log z$ is a branch of $\log$ defined on $S$. For fixed $a \in \mathbb{C}$, we define

$$a^z = e^{(\log a)z} \quad \text{and} \quad z^a = e^{a \log z}$$

For these particular branches of $\log$, then these functions are differentiable on $S$ with derivatives

$$\frac{d}{dz} a^z = (\log a) e^{(\log a)z} = (\log a) a^z$$

$$\frac{d}{dz} z^a = a^z \frac{e^{a \log z} - e^{a \log z}}{z} = a^z \frac{e^{a \log z} - e^{a \log z}}{z} = a^z \frac{e^{a \log z} - e^{a \log z}}{z} = a^z \frac{e^{a \log z} - e^{a \log z}}{z}$$

**Proof:** Chain rule.

So $z^a$ is differentiable when we fix a branch of $\log$ which makes it single-valued.

**Trig Functions**

In your homework, you'll come up with an alternate representation of $\sin(z)$ and $\cos(z)$ for all $z \in \mathbb{C}$. Also, have one for $\sinh(z)$ and $\cosh(z)$, easy to show these are entire and the following properties hold.
**Prop 11.3**: \( \sin(z), \cos(z), \sinh(z), \cosh(z) \) ARE ENTIRE AND

\[ a) \frac{d}{dz} \sin z = \cos z \quad \frac{d}{dz} \cos z = -\sin z \]

\[ b) \sin^2 z + \cos^2 z = 1 \]

\[ c) \frac{d}{dz} \sinh(z) = \cosh(z) \quad \text{AND} \quad \frac{d}{dz} \cosh(z) = \sinh(z) \]

\[ d) \cosh^2 z - \sinh^2 z = 1 \]

**Proof**: 

**Proof of (d)**: 

\[
\left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \left(\frac{e^{2z} + 2 + e^{-2z}}{4}\right) - \left(\frac{e^{2z} - 2 + e^{-2z}}{4}\right) = 1
\]

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**Contour Integration**: WE WANT TO START INTEGRATING COMPLEX FUNCTIONS (NOT JUST ANALYTIC ONS), AND TO DO THIS WE WILL INTEGRATE ALONG PATHS.

For \( f: (a,b) \rightarrow \mathbb{C} \), WHERE \( f(t) = u(t) + iv(t) \) THEN

\[
\int_a^b f(t) \, dt = \int_a^b u(t) \, dt + i \int_a^b v(t) \, dt
\]